On Equivalence of Differential Equations

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To Ulo Lumiste on his 70th birthday

Abstract. In this paper we investigate the equivalence of nonlinear partial differential equations. We consider PDEs as submanifolds of the jet bundles and study the restriction of the Cartan and the metasymplectic structures on the equation. They generate curvature-like invariants of the differential equation. We formulate our equivalence theorem in terms of these invariants.

0. Introduction

In this paper we propose some scheme to investigate the problem of equivalence of non linear partial differential equations. We consider such equations (PDEs) as a subbundles of the jet bundles $\pi_k : J^k\pi \to M$ where $\pi : E \to M$ is a smooth bundle.

Given two PDEs $\mathcal{E}$ and $\mathcal{E}'$ we call *an (external) equivalence* the prolongation $\varphi^{(k)}$ of any fiberwise diffeomorphism $\varphi : E \to E'$, which maps $\mathcal{E}$ to $\mathcal{E}'$. If $\dim\pi = \dim\pi' = 1$ the equivalence mappings $\varphi^{(k)}$ are prolongations of contact transformations $\varphi^{(1)} : J^1\pi \to J^1\pi'$, which can be non-projectable to the maps $\varphi : E \to E'$. By an *internal equivalence* we mean a diffeomorphism $\Phi : \mathcal{E} \to \mathcal{E}'$, which preserves the Cartan distribution.

Given such a concept two problems appear. First there goes a Lie-Backlund problem of whether a given internal equivalence $\Phi : \mathcal{E} \to \mathcal{E}'$ comes from a mappings of the bundles downstairs $\varphi : E \to E'$ or, in the case when $\dim\pi = 1$, $\varphi^{(1)} : J^1\pi \to J^1\pi'$, i.e. $\Phi = \varphi^{(k)}$? Upon the trivial equation $\mathcal{E} = J^k\pi$ the corresponding fact is well-known and is called Lie-Backlund theorem, for the differential equations case see [**4**].

We will use this theorem in equaling metasymplectic and special metasymplectic transportations. The exceptional case ($\dim\pi = \dim\pi' = 1$) can be treated similarly to the general case, when all equivalences are lifted from the zero-level jets $\varphi : E \to E'$. We however do not study this case separately, leaving the parallel theory aside.

Next problem is the question of prolongation existence. Actually with every equation one associates its prolongations $\mathcal{E}^{(s)} \subset J^{k+s}\pi$ which in many cases can be considered as equations itself. Thus every equivalence φ or $\varphi^{(1)}$ gives rise to a sequence of equation morphisms $\varphi^{(k+s)} : \mathcal{E}^{(s)} \to \mathcal{E}'^{(s)}$.

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To investigate this situation formally we introduce a notion of framed k-jet of isomorphism, that is a k-jet of a bundle isomorphism together with metasymplectic transformations of corresponding Cartan subspaces. In contrast to k-jets of bundle isomorphisms we can prolong framed ones. If such isomorphisms in addition map $\mathcal{E}$ to $\mathcal{E}'$, then they establish a formal equivalence of the differential equations. We apply this construction to formally integrable equations and find first obstructions for formal classification.

For non integrable differential equations the picture is slightly different. Namely, we introduce Weyl tensors of PDEs (see [**11**] for details) and show that lifting of framed k-jet of isomorphism to the next prolongation is possible if and only if it preserves the Weyl tensors. This result shows that equivalence of the Weyl tensors is a necessary condition to passing from PDEs equivalency on level of k-jets to $(k+1)$-jet level.

These results can be applied to both differential equations ([**5, 6**]) and geometrical structures equivalence problem. The last are operated via the approach proposed in [**10, 11**] and we will give a special consideration to them elsewhere.

1. Cartan distributions, metasymplectic structures and curvatures

In this section we recall basic facts concerning to the jet space geometry (see [**1**], [**4**], [**9**] for details). They originate from the classical works of Lie ([**8**]) and Cartan ([**2**]).

1.1. Curvatures of distributions. Let M be a smooth manifold and let $P : a \in M \longmapsto P(a) \subset \tau_a$ be a distribution. We denote by $C^\infty(P)$ the $C^\infty(M)$-module of vector fields which are tangent to P and by $P_0 \subset \Omega^1(M)$ the annihilator of P.

Let us fix a point $a \in M$. Then de Rham differential induces a map

$$\xi : \omega \in P_0 \longmapsto d\omega|_{P(a)},$$

which satisfies the following property

$$\xi(f\omega) = f(a)\,\xi(\omega)$$

for all $f \in C^\infty(M)$.

Taking the value of ξ at the point we get the following linear map

$$\Xi_a : P_0(a) \to \Lambda^2(P^*(a))$$

or tensor

$$\Xi_a \in \Lambda^2(P^*(a)) \otimes \nu_a,$$

where

$$\nu_a = P_0^*(a) = \tau_a \diagup P(a)$$

is the normal space to the subspace $P(a) \subset \tau_a$.

Definition 1. *The tensor $\Xi \in \Omega^2(P^*) \otimes \nu$, where $\Xi : a \longmapsto \Xi_a$, is called* curvature *of distribution P.*

Remark 1. *One could consider the curvature tensor as a 2-form on the distribution which takes values in the normal bundle.*

It is easy to see that this pointwise construction admits more invariant form.

PROPOSITION 1. *De Rham differential induces $C^{\infty}(M)$-linear map*

$$\begin{aligned} d_P &: P_0 \to \Omega^2(M) \diagup P_0 \wedge \Omega^1(M), \\ d_P &: \omega \longmapsto d\omega \bmod P_0 \wedge \Omega^1(M). \end{aligned}$$

Then $\Omega^2(M) \diagup P_0 \wedge \Omega^1(M) = \Omega^2(P^*)$ and d_P coincides with Ξ.

Let $\phi : M \to N$ be a (local) diffeomorphism, and let P and Q be distributions on M and N respectively, $\dim P = \dim Q$.

If ϕ maps one distribution to another: $\phi_*(P(a)) = Q(b)$, $b = \phi(a)$, then the differential $\phi_* : \tau_a \to \tau_b$ induces the normal (quotient) linear isomorphism $\phi_*^v : \nu_a \to \nu_b$.

We define the image $\phi^*(\Xi_b)$ under this map from the following commutative diagram

$$\begin{tikzcd} \nu_a^* \arrow[r, "\phi^*(\Xi_b)"] & \Lambda^2(P^*(a)) \\ \nu_b^* \arrow[u, "(\phi^v)^*"] \arrow[r, "\Xi_b"] & \Lambda^2(P^*(b)). \arrow[u, "\Lambda^2(\phi)^*"'] \end{tikzcd}$$

DEFINITION 2. *We say that 1-jet of local diffeomorphism $\phi : M \to N$ is an isomorphism of the distributions at $a \in M$ if $\phi_*(P(a)) = Q(b)$ and $\phi^*(\Xi_b) = \Xi_a$.*

1.2. Cartan distributions. Let $\pi : E \to M$ be a smooth bundle.

We denote by $[s]_a^k = j_k s(a)$ the k-jet of a section $s : M \to E$ at the point $a \in M$, and by $J_a^k \pi$ the space of all k-jets at the point.

The space of all k-jets at all points, as usual, we shall denote by $J^k \pi = \bigcup_{a \in M} J_a^k \pi$ and by $\pi_{k,l} : J^k \pi \to J^l \pi$, $k > l$, we shall denote the natural projections.

Note that $J^0 \pi = E$.

For any point $a_k \in J^k \pi$ we denote by $a_l = \pi_{k,l}(a_k) \in J^l \pi$ the corresponding projections and by $F(a_{k-1}) = \pi_{k,k-1}^{-1}(a_{k-1})$ the fibre of $\pi_{k,k-1}$ over a_{k-1}, $F(a) = \pi^{-1}(a)$.

It is well known (see, for example, [**3**], [**4**] or [**1**]) that $F(a_{k-1})$ has a natural (with respect to automorphisms of the bundle) affine structure when $k \geq 1$, and the corresponding vector space is $S^k \tau_a^* \otimes \nu_{a_0}$ where $\tau_a^* = T_a^* M$ is the cotangent space and $\nu_{a_0} = T_{a_0}\left(\pi^{-1}(a)\right)$ is the tangent space to the fibre.

Any local section $s : M \to E$ of π produces the section $j_k s : M \to J^k \pi$ of the bundle π_k where $j_k s : a \in M \longmapsto [s]_a^k \in J^k \pi$. The tangent space $T_{a_k}(j_k s(M))$ at the point $a_k = [s]_a^k$ depends on the element $a_{k+1} = [s]_a^{k+1}$ only. Denote this space by $L(a_{k+1})$.

The *Cartan space* $C(a_k)$ is the vector subspace of $T_{a_k}\left(J^k \pi\right)$ generated by $L(a_{k+1})$ for all $a_{k+1} \in F(a_k)$.

The *Cartan distribution* $\mathcal{C}$ on $J^k \pi$ is the distribution generated by the Cartan spaces:

$$\mathcal{C} : J^k \pi \ni a_k \longmapsto C(a_k) \subset T_{a_k}\left(J^k \pi\right).$$

It follows from the construction that $(\pi_{k,k-1})_* : C(a_k) \to L(a_k)$, and it can be shown ([**4**]) that $C(a_k) = (\pi_{k,k-1})_*^{-1}(L(a_k))$.

In other words, any point $a_{k+1} \in F(a_k)$ produces the following splitting

$$C(a_k) = T_{a_k}(F(a_{k-1})) \oplus L(a_{k+1}) = S^k \tau_a^* \otimes \nu_{a_0} \oplus L(a_{k+1}). \tag{1.1}$$

By the very definition the graphs $(j_k s(M)) \subset J^k \pi$ are integral manifolds of the Cartan distribution. The significance of the distribution descends from its following property: every integral manifold L of the distribution of dimension $m = \dim M$ for which the projection $\pi_k : L \to M$ is a diffeomorphism has the form $j_k s(M)$.

Let $\mathcal{E} \subset J^k \pi$ be a partial differential equation. We denote by $C_{\mathcal{E}}$ the restriction of the Cartan distribution on $\mathcal{E}$:

$$C_{\mathcal{E}}(a_k) = C(a_k) \cap T_{a_k}(\mathcal{E})$$

for all $a_k \in \mathcal{E}$.

In this case the above splitting takes the following form

$$C_{\mathcal{E}}(a_k) = g(a_k) \oplus L(a_{k+1}),$$

where $a_{k+1} \in \mathcal{E}^{(1)}$ is an element of the 1-st prolongation, and

$$g(a_k) = T_{a_k}(F(a_{k-1})) \cap T_{a_k}(\mathcal{E})$$

is the symbol of $\mathcal{E}$ at point a_k.

1.3. Cartan forms and metasymplectic structures. Let $\Omega_0^k(J^k\pi)$ be the $C^\infty(J^k\pi)$-module of π_k-horizontal forms on $J^k\pi$.

Define the operator

$$\widehat{d} : \Omega_0^k(J^k\pi) \to \Omega_0^{k+1}(J^{k+1}\pi)$$

by the following universal property:

$$j_{k+1}^* s\left(\widehat{d}\omega\right) = d\left(j_k^* s(\omega)\right)$$

for all $s \in C^\infty(\pi)$. Here $\omega \in \Omega_0^k(J^k\pi)$.

Then

$$\widehat{d}(\alpha \wedge \beta) = \widehat{d}(\alpha) \wedge \pi_{k+1,k}^*(\beta) + (-1)^{|\alpha|} \pi_{k+1,k}^*(\alpha) \wedge \widehat{d}(\beta)$$

and

$$\widehat{d}^2 = 0.$$

The *Cartan 1-form* $U(f)$ corresponding to function $f \in C^\infty(J^{k-1}\pi)$ is the following differential 1-form on the space $J^k\pi$:

$$U(f) = \pi_{k,k-1}^*(df) - \widehat{d}(f) \in \Omega^1(J^k\pi).$$

To compute the value of the differential 1-form on a vector $X \in T_{a_k}(J^k\pi)$ we use the decomposition

$$\begin{aligned} T_{a_{k-1}}(J^{k-1}\pi) &= T_{a_{k-1}}^v \oplus L(a_k); \\ X &= X^v + X^h \end{aligned}$$

where $T_{a_{k-1}}^v = T_{a_{k-1}}(J_a^{k-1}\pi)$.

Then

$$U(f)(X) = \langle df, ((\pi_{k,k-1})_* X)^v \rangle.$$

Moreover, the Cartan forms $U(f)$, $f \in C^\infty(J^{k-1}\pi)$ determine the Cartan distribution on $J^k\pi$ ([**4**]).

Let us describe now the curvature tensor of the Cartan distribution.

In our case differentials of the Cartan forms determine a linear operator

$$\Omega_{a_k} : C^{\infty}\left(J^{k-1}\pi\right) \to \Lambda^2\left(C^*\left(a_k\right)\right),$$

where

$$\Omega_{a_k}(f) = dU(f)|_{C(a_k)}.$$

Moreover, it is easy to see that Ω_{a_k} is a derivation, and therefore we can look at Ω_{a_k} as a linear operator

$$\Omega_{a_k} : T^*_{a_{k-1}}\left(J^{k-1}\pi\right) \to \Lambda^2\left(C^*\left(a_k\right)\right).$$

On the other hand, $\Omega_{a_k}(f) = 0$ if $f \in \pi^*_{k-1,k-2}\left(C^{\infty}\left(J^{k-2}\pi\right)\right)$.

Consider the following exact sequence

$$0 \to T^*_{a_{k-2}}\left(J^{k-2}\pi\right) \to T^*_{a_{k-1}}\left(J^{k-1}\pi\right) \to S^{k-1}\tau_a \otimes \nu^*_{a_0} \to 0.$$

Operator Ω_{a_k} vanishes on $T^*_{a_{k-2}}\left(J^{k-2}\pi\right)$ and therefore it can be considered as a linear operator on the factor space.

We denote this operator by

$$\Omega_{a_k} : S^{k-1}\tau_a \otimes \nu^*_{a_0} \to \Lambda^2\left(C^*\left(a_k\right)\right)$$

also and will call *metasymplectic structure* on Cartan space $C(a_k)$ ([**9**]).

By the definition both $U_{a_k}(f)$ and $\Omega_{a_k}(\lambda)$, $\lambda \in S^{k-1}\tau_a \otimes \nu^*_{a_0}$, vanish on tangent planes of integral manifolds of the Cartan distribution. Since $F(a_{k-1})$ is also integral then according to splitting (1.1) to compute the metasymplectic structure $\Omega_{a_k}(\lambda)$ it is enough to determine the value $\Omega_{a_k}(\lambda)$ on bivectors of the form $\theta \wedge X$ where $\theta \in S^k\tau^*_a \otimes \nu_{a_0}$ and $X \in L(a_{k+1}) \approx \tau_a$. This can be done according to [**9**] by means of the Spencer operator $\delta : S^k\tau^*_a \otimes \nu_{a_0} \to \tau^*_a \otimes S^{k-1}\tau^*_a \otimes \nu_{a_0}$:

$$\Omega_{a_k}(\lambda)(X,\theta) = \langle \lambda, i_X\delta\theta \rangle. \tag{1.2}$$

1.4. Cartan and Bott connections.

Definition 3. 1. *An m-dimensional ($m = \dim M$) subspace $H(a_k) \subset C(a_k)$ is called* horizontal *if $(\pi_k)_*(H(a_k)) = \tau_a$.*

2. *A subdistribution of the Cartan distribution $H : a_k \in J^k\pi \longmapsto H(a_k) \subset C(a_k)$ consisting of horizontal subspaces is called* Cartan connection *on π_k.*

There is a simple way to get a Cartan connection. Namely, let $\Psi : J^k\pi \to J^{k+1}\pi$ be a section of the bundle $\pi_{k+1,k}$. Then $H_\Psi(a_k) = L(\Psi(a_k))$ is obviously a Cartan connection.

Definition 4. *A Cartan connection of the type H_Ψ is called* Bott connection.

Let us fix a Bott connection H_Ψ. Then any other Cartan connection H can be viewed as a graph of smooth family σ of operators $\sigma_{a_k} : L(\Psi(a_k)) \to T_{a_k}(F(a_{k-1}))$:

$$H(a_k) = \operatorname{graph}\left(\sigma_{a_k} : L(\Psi(a_k)) \to T_{a_k}(F(a_{k-1}))\right).$$

Using isomorphisms $L(\Psi(a_{k+1})) \approx \tau_a$ and $T_{a_k}(F(a_{k-1})) \approx S^k\tau^*_a \otimes \nu_{a_0}$ we can represent σ in the form

$$\sigma_{a_k} : \tau_a \to S^k\tau^*_a \otimes \nu_{a_0} \quad \text{or} \quad \sigma_{a_k} \in \tau^*_a \otimes S^k\tau^*_a \otimes \nu_{a_0}.$$

Proposition 2. *Cartan connection $H = H_\sigma$ is a Bott connection if and only if $\delta\sigma = 0$ where $\delta : \tau_a^* \otimes S^k\tau_a^* \otimes \nu_{a_0} \to \Lambda^2(\tau_a^*) \otimes S^{k-1}\tau_a^* \otimes \nu_{a_0}$ is the Spencer δ-operator.*

Proof. H is a Bott connection if and only if all spaces $H(a_k)$ are isotropic with respect to the metasymplectic structure. In other words, if

$$\langle \lambda, i_{X_2}(\sigma(X_1)) - i_{X_1}(\sigma(X_2)) \rangle = 0$$

for all $\lambda \in S^{k-1}\tau_a \otimes \nu_{a_0}^*$, and $X_1, X_2 \in \tau_a$.

The last condition obviously means $\delta\sigma = 0$. □

Let $\mathcal{E} \subset J^k\pi$ be a differential equation and $\mathcal{C}_\mathcal{E}$ be the restriction of the Cartan distribution on $\mathcal{E}$.

Definition 5. *A* Cartan connection *on PDE $\mathcal{E}$ is a smooth field H of horizontal spaces on $\mathcal{E}$ such that $H(a_k) \subset C_\mathcal{E}(a_k)$ for all $a_k \in \mathcal{E}$. A Cartan connection on $\mathcal{E}$ is called a* Bott connection *if all horizontal subspaces $H(a_k)$ are isotropic with respect to the metasymplectic structure.*

Remark 2. *A Bott connection can be identified with a section $\Psi : \mathcal{E} \to \mathcal{E}^{(1)}$.*

Definition 6. *Let H be a Cartan connection on PDE $\mathcal{E}$. The tensor $\Omega_H(a_k) = \Omega_{a_k}|_{H(a_k)}$ is called a* curvature *of the Cartan connection.*

As above we have the following description of the Bott connections in terms of the curvature.

Proposition 3. *A Cartan connection H on PDE $\mathcal{E}$ is a Bott connection if and only if $\Omega_H = 0$.*

Remark that the curvature $\Omega_H(a_k)$ can be also considered as a linear mapping

$$\Omega_H(a_k) : S^{k-1}\tau_a \otimes \nu_{a_0}^* / \operatorname{Ann} g(a_{k-1}) \longrightarrow \Lambda^2(\tau_a^*)$$

where $g(a_{k-1}) = T_{a_{k-1}}(\mathcal{E}_{k-1} \cap F(a_{k-2}))$ and $\mathcal{E}_{k-1} = \pi_{k,k-1}(\mathcal{E})$.

In other words,

$$\Omega_H(a_k) \in \Lambda^2(\tau_a^*) \otimes g(a_{k-1}).$$

1.5. Lie equations of symmetries. For given differential equation $\mathcal{E}$ we associate the Lie algebra of its symmetries $\operatorname{Sym}\mathcal{E}$ consisting of π-projectable vector fields X on E with prolongations $X^{(k)}$ being tangent to $\mathcal{E}$. This Lie algebra is the space of solutions of the equation $\operatorname{Lie}\mathcal{E}$ for vector fields X ([**7**]). Lie pseudogroup corresponding to $\operatorname{Lie}\mathcal{E}$ consists of autoequivalences of $\mathcal{E}$.

For a pair of equations $\mathcal{E} \subset J^k\pi$ and $\mathcal{E}' \subset J^k\pi'$ we can consider also the set of all equivalences, that is bundle isomorphisms $\phi : E \to E'$ such that $\phi^{(k)}(\mathcal{E}) = \mathcal{E}'$. This set can be represented as the set of solutions of PDE $\operatorname{Lie}(\mathcal{E},\mathcal{E}')$ where $\operatorname{Lie}(\mathcal{E},\mathcal{E}') \subset J^k(\pi,\pi')$ consists of k-jets $[\phi]_a^k \in J_a^k(\pi,\pi')$ of bundle isomorphisms such that $\phi^{(k)}(\mathcal{E}_a) = \mathcal{E}'_a$. Here $\mathcal{E}_a = \mathcal{E} \cap \pi_k^{-1}(a)$.

We shall consider the prolongations $\operatorname{Lie}^{(l)}(\mathcal{E},\mathcal{E}') \subset J^{k+l}(\pi,\pi')$ just as in usual theory of PDEs.

In this paper we are interested in the prolongation existence over some point $[\phi]_a^{k+l} \in \operatorname{Lie}^{(l)}(\mathcal{E},\mathcal{E}')$, and in the question of calculating

$$\pi_{k+s+l,k+l}\left(\operatorname{Lie}^{(l+s)}(\mathcal{E},\mathcal{E}')\right) \subset \operatorname{Lie}^{(l)}(\mathcal{E},\mathcal{E}').$$

We will consider mostly $s = 1$ since then one can prolong inductively. In such setting general results on formal integrability can be applied. We however will treat the problem differently by detecting obstructions directly along their construction.

2. Metasymplectic transformations

In this section we consider the problem of construction of formal equivalences between PDEs. So many results have local or even pointwise nature.

2.1. Prolongations of bundle isomorphisms. Let $\phi : E \to E'$ be an isomorphism of smooth bundles π and π', i.e. ϕ is a diffeomorphism of the manifolds and there exists a diffeomorphism $\overline{\phi} : M \to M'$ such that the following diagram

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "\overline{\phi}"] & M' \end{tikzcd}$$

commutes.

We define the k-th *prolongation* $\phi^{(k)} : J^k\pi \to J^k\pi'$ as follows

$$\phi^{(k)}\left([s]_a^k\right) = \left[\phi \circ s \circ \overline{\phi}^{-1}\right]_{a'}^k$$

where $a' = \overline{\phi}\,(a)$.

Then $\phi^{(k)}$ is obviously an isomorphism of bundles π_k and π'_k over diffeomorphism $\overline{\phi} : M \to M'$ and it satisfies the following properties.

THEOREM 1. 1. *Diagrams*

$$\begin{tikzcd} J^k\pi \arrow[r, "\phi^{(k)}"] \arrow[d, "\pi_{k,l}"'] & J^k\pi' \arrow[d, "\pi'_{k,l}"] \\ J^l\pi \arrow[r, "\phi^{(l)}"] & J^l\pi' \end{tikzcd}$$

commute for all $k > l$.

2. $\phi^{(k)}$ *preserves the Cartan forms* U *:*

$$U\left(\left(\phi^{(k-1)}\right)^* f\right) = \left(\phi^{(k)}\right)^* (U\,(f))$$

for all $f \in C^\infty\left(J^{k-1}\pi'\right)$.

3. $\phi^{(k)}$ *preserves the metasymplectic structure* Ω, *i.e. the following diagrams*

$$\begin{tikzcd} S^{k-1}\tau_{a'} \otimes \nu^*_{a'_0} \arrow[r, "\Omega'_k"] \arrow[d, "S^{k-1}\overline{\phi}_*^{\,-1} \otimes \phi^*"'] & \Lambda^2\left(C^*\left(a'_k\right)\right) \arrow[d, "\Lambda^2(\phi^{(k)})^*"] \\ S^{k-1}\tau_a \otimes \nu^*_{a_0} \arrow[r, "\Omega_k"] & \Lambda^2\left(C^*\left(a_k\right)\right) \end{tikzcd}$$

commute.

We begin the proof of the theorem with the following result.

LEMMA 1. *We have*

$$\left(\phi^{(k)}\right)^*\left(\widehat{d}f\right)=\widehat{d}\left(\left(\phi^{(k-1)}\right)^*(f)\right)$$

for all $f\in C^\infty\left(J^{k-1}\pi'\right)$.

PROOF OF THE LEMMA. At first, we note that differential 1-form $(\phi^{(k)})^*\left(\widehat{d}f\right)$ is π_k-horizontal.

On the other hand, if $s'_\phi=\phi\circ s\circ\overline{\phi}^{-1}$ we get $\phi^{(k)}\circ j_k(s)=j_k\left(s'_\phi\right)\circ\overline{\phi}$, and

$$\begin{aligned}
&j_k^*(s)\left(\left(\phi^{(k)}\right)^*\left(\widehat{d}f\right)\right)=\left(\phi^{(k)}\circ j_k(s)\right)^*\left(\widehat{d}f\right)=\left(j_k\left(s'_\phi\right)\circ\overline{\phi}\right)^*\left(\widehat{d}f\right)=\\
&\overline{\phi}^*\left(j_k^*\left(s'_\phi\right)\left(\widehat{d}f\right)\right)=\overline{\phi}^*\left(d\left(j_{k-1}^*\left(s'_\phi\right)f\right)\right)=d\left(\overline{\phi}^*\left(j_{k-1}^*\left(s'_\phi\right)f\right)\right)=\\
&d\left(\left(j_{k-1}\left(s'_\phi\right)\circ\overline{\phi}\right)^*f\right)=d\left(\left(\phi^{(k-1)}\circ j_{k-1}(s)\right)^*f\right)=\\
&d(j_{k-1}^*(s)\left(\left(\phi^{(k-1)}\right)^*f\right))=j_k^*(s)\left(\widehat{d}\left(\left(\phi^{(k-1)}\right)^*f\right)\right)
\end{aligned}$$

for all $s\in C^\infty(\pi)$.

Therefore $\left(\phi^{(k)}\right)^*\left(\widehat{d}f\right)=\widehat{d}\left(\left(\phi^{(k-1)}\right)^*(f)\right)$. ☐

PROOF OF THE THEOREM. We have $U(f)=\pi_{k,k-1}^*(df)-\widehat{d}f$, and therefore

$$\begin{aligned}
&\left(\phi^{(k)}\right)^*(U(f))=\left(\phi^{(k)}\right)^*\left(\pi_{k,k-1}^*(df)\right)-\left(\phi^{(k)}\right)^*\left(\widehat{d}f\right)=\\
&\pi_{k,k-1}^*\left(d\left(\left(\phi^{(k-1)}\right)^*f\right)\right)-\widehat{d}\left(\left(\phi^{(k-1)}\right)^*(f)\right)=U\left(\left(\phi^{(k-1)}\right)^*f\right).
\end{aligned}$$

From this property of U we obtain the following commutative diagram

$$\begin{tikzcd}
C^\infty\left(J^{k-1}\pi\right) \arrow[r, "\Omega"] & \Lambda^2\left(C^*(a_k)\right) \\
C^\infty\left(J^{k-1}\pi'\right) \arrow[r, "\Omega'"] \arrow[u, "\left(\phi^{(k-1)}\right)^*"] & \Lambda^2\left(C^*(a'_k)\right) \arrow[u, "\Lambda^2\left(\left(\phi^{(k)}\right)^*_{a_k}\right)"']
\end{tikzcd}$$

for the metasymplectic structure Ω.

Commutativity of the following diagram

$$\begin{tikzcd}
J^k\pi \arrow[r, "\phi^{(k)}"] \arrow[d, "\pi_{k,k-1}"'] & J^k\pi' \arrow[d, "\pi'_{k,k-1}"] \\
J^{k-1}\pi \arrow[r, "\phi^{(k-1)}"] & J^{k-1}\pi'
\end{tikzcd}$$

gives commutativity of the differentials

$$\begin{tikzcd}
0 \arrow[r] & T^*_{a_{k-2}}\left(J^{k-2}\pi\right) \arrow[r] & T^*_{a_{k-1}}\left(J^{k-1}\pi\right) \arrow[r] & S^{k-1}\tau_a \otimes \nu^*_{a_0} \arrow[r] & 0 \\
0 \arrow[r] & T^*_{a'_{k-2}}\left(J^{k-2}\pi'\right) \arrow[r] \arrow[u, "\left(\phi^{(k-2)}\right)^*"] & T^*_{a'_{k-1}}\left(J^{k-1}\pi'\right) \arrow[r] \arrow[u, "\left(\phi^{(k-1)}\right)^*"] & S^{k-1}\tau_{a'} \otimes \nu^*_{a'_0} \arrow[r] \arrow[u, "S^{k-1}\left(\overline{\phi}_*\right)^{-1} \otimes \phi^*"'] & 0
\end{tikzcd}$$

and thus (1.2) together with the definition implies the claim. □

In the same way we get the following result on differential equations equivalences.

Theorem 2. *Let $\phi : E \to E'$ be an equivalence of differential equations $\mathcal{E} \subset J^k\pi$ and $\mathcal{E}' \subset J^k\pi'$, i.e. $\phi^{(k)}(\mathcal{E}) = \mathcal{E}'$, and let $\Omega_{\mathcal{E}} \in \Lambda^2\left(C^*_{\mathcal{E}}(a_k)\right) \otimes g_{k-1}(a_{k-1})$ and $\Omega_{\mathcal{E}'} \in \Lambda^2\left(C^*_{\mathcal{E}'}(a'_k)\right) \otimes g_{k-1}\left(a'_{k-1}\right)$, $\phi^{(k)}(a_k) = a'_k$ be the corresponding metasymplectic structures. Then $\left(\phi^{(k)}\right)^*(\Omega_{\mathcal{E}'}) = \Omega_{\mathcal{E}}$, where $\left(\phi^{(k)}\right)^*(\Omega_{\mathcal{E}'})$ is given by the following commutative diagram*

$$\begin{tikzcd}
\left(g_{k-1}(a'_{k-1})\right)^* \arrow[r, "\Omega_{\mathcal{E}'}"] \arrow[d, "S^{k-1}(\overline{\phi}_*)^{-1} \otimes \phi^*"] & \Lambda^2\left(C^*_{\mathcal{E}'}(a'_k)\right) \arrow[d, "\Lambda^2(\phi^{(k)})^*"] \\
\left(g_{k-1}(a_{k-1})\right)^* \arrow[r, "\left(\phi^{(k)}\right)^*(\Omega_{\mathcal{E}'})"] & \Lambda^2\left(C^*_{\mathcal{E}}(a_k)\right).
\end{tikzcd}$$

2.2. Metasymplectic transformations. Let $\psi : J^k\pi \to J^k\pi'$ be a smooth map. 1-jet of ψ at $a_k \in J^k\pi$ is determined by the image $a'_k \in J^k\pi'$ of the point and a linear map $\Phi : T_{a_k}\left(J^k\pi\right) \to T_{a'_k}\left(J^k\pi'\right)$. In the case when $\psi = \phi^{(k)}$ is an equivalence Φ maps the Cartan space to the primed Cartan space. Moreover this map is $\pi_{k,k-1}$-vertical and preserves the metasymplectic form.

Thus we are given the following object $(a_k, a'_k; \Phi)$ where the second term stands for a linear isomorphism $\Phi : C(a_k) \to C(a'_k)$.

Definition 7.
1. *A linear isomorphism $\Phi : C(a_k) \to C(a'_k)$ is called* metasymplectic transformation *if it maps isotropic subspaces to isotropic ones.*
2. *A metasymplectic transformation $\Phi : C(a_k) \to C(a'_k)$ is called* special *if Φ is a vertical isomorphism, that is, Φ preserves the tangent spaces to $\pi_{k,k-1}$-fibres and induced the vertical part $\Phi_v : S^k\tau^*_a \otimes \nu_{a_0} \to S^k\tau^*_{a'} \otimes \nu_{a'_0}$ of the map.*

Remark 3. *When $k = 1$ and $\dim\pi = 1$ metasymplectic transformations are simply conformal symplectic transformations. If $k \geq 2$ or $\dim\pi \geq 2$ there is no difference between metasymplectic and special metasymplectic transformations.*

We investigate now the structure of (special) metasymplectic transformations.

Let $\Phi : C(a_k) \to C(a'_k)$ be a special metasymplectic transformation. We define a *horizontal part* of Φ as a linear isomorphisms $\Phi_h : \tau_a \to \tau_{a'}$ given by the following commutative diagram

$$\begin{tikzcd} C\left(a_k\right) \arrow[r, "\Phi"] \arrow[d, "\left(\pi_k\right)_*"'] & C\left(a'_k\right) \arrow[d, "\left(\pi'_k\right)_*"] \\ \tau_a \arrow[r, "\Phi_h"] & \tau_{a'} . \end{tikzcd}$$

LEMMA 2. *Φ_h is well defined.*

PROOF. It follows from the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & S^k\tau_a^* \otimes \nu_{a_0} \arrow[r] \arrow[d, "\Phi_v"'] & C\left(a_k\right) \arrow[r, "\pi_{k*}"] \arrow[d, "\Phi"'] & \tau_a \arrow[r] \arrow[d, "\Phi_h"'] & 0 \\ 0 \arrow[r] & S^k\tau_{a'}^* \otimes \nu_{a'_0} \arrow[r] & C\left(a'_k\right) \arrow[r, "\pi'_{k*}"] & \tau_{a'} \arrow[r] & 0. \end{tikzcd}$$

□

THEOREM 3. *Let $\Phi : C\left(a_k\right) \to C\left(a'_k\right)$ be a special metasymplectic transformation. There is a linear isomorphism $\phi : \nu_{a_0} \to \nu_{a'_0}$ such that*

$$\Phi_v = S^k\left(\Phi_h^{-1}\right)^* \otimes \phi : S^k\tau_a^* \otimes \nu_{a_0} \to S^k\tau_{a'}^* \otimes \nu_{a'_0}.$$

PROOF. Let $a_{k+1} \in F\left(a_k\right)$. Denote by $\overline{X} \in L\left(a_{k+1}\right)$ the image of vector $X \in \tau_a$ under the isomorphism $\left(\pi_k\right)_* : L\left(a_{k+1}\right) \to \tau_a$.

Take the following isotropic and decomposable pair $\left(\overline{X}, \theta = p^k \otimes v \in S^k\tau_a^* \otimes \nu_{a_0}\right)$ of vectors of $C\left(a_k\right)$ where $p \in \tau_a^*$ and $\langle X, p\rangle = 0$.

Then the pair $\left(\Phi\left(\overline{X}\right), \Phi_v\left(p^k \otimes v\right)\right) = \left(\overline{\Phi_h\left(X\right)}, \Phi_v\left(p^k \otimes v\right)\right)$ is isotropic too.

Here we also denoted by $\overline{Y} \in L\left(a'_{k+1}\right)$ the image of vector $Y \in \tau_{a'}$ under the isomorphism $\left(\pi'_k\right)_* : L\left(a'_{k+1}\right) \to \tau_{a'}$, where $L\left(a'_{k+1}\right) = \Phi\left(L\left(a_{k+1}\right)\right)$.

Therefore, tensor $\Phi_v\left(p^k \otimes v\right) \in S^k\tau_{a'}^* \otimes \nu_{a'_0}$ degenerates along directions $\Phi_h\left(X\right)$ every time when $\langle X, p\rangle = 0$.

Hence, $\Phi_v\left(p^k \otimes v\right) = \left(\left(\Phi_h^{-1}\right)^*(p)\right)^k \otimes \phi\left(v\right)$ for some linear isomorphism $\phi : \nu_{a_0} \to \nu_{a'_0}$. □

COROLLARY 1. *Any special metasymplectic transformation preserves the metasymplectic structure.*

2.3. Metasymplectic structures over differential equations. In this section we introduce and investigate metasymplectic transformations over differential equations.

Let $\mathcal{E} \subset J^k\pi$ and $\mathcal{E}' \subset J^k\pi'$ be differential equations, and $a_k \in \mathcal{E}$, $a'_k \in \mathcal{E}'$ be fixed points.

We will assume here that $\pi_k : \mathcal{E} \to M$ and $\pi'_k : \mathcal{E}' \to M'$ are surjections.

DEFINITION 8. 1. *A linear isomorphism $\Phi : C_{\mathcal{E}}\left(a_k\right) \to C_{\mathcal{E}'}\left(a'_k\right)$ is called* metasymplectic transformation *if it maps isotropic subspaces to isotropic ones.*

2. *A metasymplectic transformation* $\Phi : C_{\mathcal{E}}(a_k) \to C_{\mathcal{E}'}(a'_k)$ *is called* special *if* Φ *is a vertical isomorphism, that is,* Φ *maps the tangent space to fibres of the projection* $\pi_{k,k-1} : \mathcal{E} \to J^{k-1}\pi$ *to the primed one. The induced map we call the* vertical part $\Phi_v : g(a_k) \to g(a'_k)$ *of the map.*

As above we define the horizontal part of Φ from the commutative diagram

$$\begin{tikzcd} C_{\mathcal{E}}(a_k) \arrow[r, "\Phi"] \arrow[d, "(\pi_k)_*"'] & C_{\mathcal{E}'}(a'_k) \arrow[d, "(\pi'_k)_*"] \\ \tau_a \arrow[r, "\Phi_h"] & \tau_{a'}. \end{tikzcd}$$

Remark 4. *There is no difference between special and metasymplectic transformations if*

$$\dim g(a_k) > n.$$

Denote by $\mathrm{Char}^{\mathbb{C}}(\mathcal{E}, a_k)$ the set of all complex characteristics of the differential equation at the point, $\mathrm{Char}^{\mathbb{C}}(\mathcal{E}, a_k) \subset \tau_a^{\mathbb{C}*} \backslash 0$.

For any $p \in \mathrm{Char}^{\mathbb{C}}(\mathcal{E}, a_k)$ we denote by $K_p \subset \nu_{a_0}^{\mathbb{C}}$ the corresponding kernel space:

$$K_p = \left\{ v \in \nu_{a_0}^{\mathbb{C}} \,\middle|\, p^k \otimes v \in g^{\mathbb{C}}(a_k) \right\}.$$

The proof of the following theorem is similar to the one of theorem 3.

Theorem 4. *Let* $\Phi : C_{\mathcal{E}}(a_k) \to C_{\mathcal{E}'}(a'_k)$ *be a special metasymplectic transformation and let* $\mathcal{E}, \mathcal{E}'$ *be such differential equations that*

1. *The* 1*-st prolongation* $\mathcal{E}^{(1)}$ *is not empty over point* $a_k \in \mathcal{E}$.
2. *Vector subspaces* K_p *generates* $\nu_{a_0}^{\mathbb{C}}$:

$$\sum_{p \in \mathrm{Char}^{\mathbb{C}}(\mathcal{E}, a_k)} K_p = \nu_{a_0}^{\mathbb{C}},$$

3. *Tensors* $p^k \otimes v$*, where* $p \in \mathrm{Char}^{\mathbb{C}}(\mathcal{E}, a_k)$ *and* $v \in K_p$*, generate* $g^{\mathbb{C}}(a_k)$.

Then the vertical part of Φ *has the form*

$$\Phi_v = S^k \left(\Phi_h^{-1}\right)^* \otimes \phi : g(a_k) \to g(a'_k)$$

for some linear isomorphism $\phi : \nu_{a_0} \to \nu_{a'_0}$.

Remark 5. *For formally integrable PDEs condition (3) of the theorem above (after some number of prolongations) can be reformulated in terms the projectivization* $\mathrm{char}^{\mathbb{C}}(\mathcal{E}, a_k)$ *of* $\mathrm{Char}^{\mathbb{C}}(\mathcal{E}, a_k)$. *Namely, this projective manifold should be non degenerated (= does not belong to hyperspaces).*

2.4. One step lifting of metasymplectic transformations. In this section we will be interested in the lifting of special metasymplectic transformations $\Phi : C(a_k) \to C(a'_k)$ to the level $(k+1)$, i.e. we will search for a pair $\left(a_{k+1}, a'_{k+1}; \widetilde{\Phi}\right)$ where $\pi_{k+1,k}(a_{k+1}) = a_k$, $\pi'_{k+1,k}(a'_{k+1}) = a'_k$ and $\widetilde{\Phi} : C(a_{k+1}) \to C(a'_{k+1})$ is such a vertical isomorphism that it preserves the metasymplectic structure, that is

$$\begin{tikzcd} S^k\tau_a \otimes \nu^*_{a_0} \arrow[r, "\Omega_{a_k}"] & \Lambda^2\left(C^*\left(a_{k+1}\right)\right) \\ S^k\tau_{a'} \otimes \nu^*_{a'_0} \arrow[r, "\Omega_{a'_k}"] \arrow[u, "(\Phi_v)^*"] & \Lambda^2\left(C^*\left(a'_{k+1}\right)\right) \arrow[u, "\Lambda^2(\widetilde{\Phi}^*)"'] \end{tikzcd}$$

and makes the natural diagram commutative:

$$\begin{tikzcd} C\left(a_{k+1}\right) \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\left(\pi_{k+1,k}\right)_*"'] & C\left(a'_{k+1}\right) \arrow[d, "\left(\pi'_{k+1,k}\right)_*"] \\ C\left(a_k\right) \arrow[r, "\Phi"] & C\left(a'_k\right). \end{tikzcd}$$

Moreover we shall investigate a relation between a_{k+1} and a'_{k+1} given by Φ.

Note that every element $a_{k+1} \in F\left(a_k\right)$ gives the splitting

$$C\left(a_k\right) = T_{a_k}\left(F\left(a_{k-1}\right)\right) \oplus L\left(a_{k+1}\right)$$

and every other element $\widetilde{a}_{k+1} \in \pi^{-1}_{k+1,k}\left(a_k\right)$ represents $L(\widetilde{a}_{k+1})$, which can be considered as the graph of the map $\sigma : L\left(a_{k+1}\right) \to T_{a_k}\left(F\left(a_{k-1}\right)\right)$ or due to the isomorphisms $T_{a_k}\left(F\left(a_{k-1}\right)\right) \approx S^k\tau^*_a \otimes \nu_{a_0}$ and $L\left(a_{k+1}\right) \approx \tau_a$ as a linear map

$$\sigma : \tau_a \to S^k\tau^*_a \otimes \nu_{a_0} \text{ or tensor } \sigma \in \tau^*_a \otimes S^k\tau^*_a \otimes \nu_{a_0}.$$

Since $L(\widetilde{a}_{k+1})$ is isotropic, σ is δ-closed. The above construction shows that it is δ-exact, i.e. σ lies in the image of the inclusion $\delta : S^{k+1}\tau^*_a \otimes \nu_{a_0} \hookrightarrow \tau^*_a \otimes S^k\tau^*_a \otimes \nu_{a_0}$.

Now let us begin to construct a pair $\left(a_{k+1}, a'_{k+1}; \widetilde{\Phi}\right)$ by the pair $\left(a_k, a'_k; \Phi\right)$.

First, since Φ preserves vertical subspaces it maps horizontal subspaces to horizontal ones.

Second, Φ preserves the metasymplectic structure. Therefore, it maps isotropic planes to isotropic ones also. Thus, the image of subspace $L\left(a_{k+1}\right)$ being horizontal and isotropic has the form $L\left(a'_{k+1}\right)$ for some $a'_{k+1} \in \pi'^{-1}_{k+1,k}\left(a'_k\right)$.

In other words, Φ determines an isomorphism $\Psi : \pi^{-1}_{k+1,k}\left(a_k\right) \to \pi'^{-1}_{k+1,k}\left(a'_k\right)$.

Let us fix a pair $\left(a_{k+1}, a'_{k+1} = \Psi\left(a_{k+1}\right)\right)$. The differential of Ψ at a_{k+1} we shall denote by $\widetilde{\Phi}_v$, it will be the vertical part of $\widetilde{\Phi}$:

$$\widetilde{\Phi}_v = \Psi_* : T_{a_{k+1}}\left(F\left(a_k\right)\right) \to T_{a'_{k+1}}\left(F\left(a'_k\right)\right).$$

To define $\widetilde{\Phi}$ on whole $C\left(a_{k+1}\right)$ we choose some horizontal planes $L\left(a_{k+2}\right)$ and $L\left(a'_{k+2}\right)$ into $C\left(a_{k+1}\right)$ and $C\left(a'_{k+1}\right)$ correspondingly.

We define the horizontal part $\widetilde{\Phi}_h$ of $\widetilde{\Phi}$ by the diagram

$$\begin{tikzcd} L\left(a_{k+2}\right) \arrow[r, "\widetilde{\Phi}_h"] \arrow[d, "\pi_{k+1,k}"'] & L\left(a'_{k+2}\right) \arrow[d, "\pi'_{k+1,k}"] \\ L\left(a_{k+1}\right) \arrow[r, "\Phi"] & L\left(a'_{k+1}\right), \end{tikzcd}$$

and

(2.1)

$$\widetilde{\Phi} = \widetilde{\Phi}_v \oplus \widetilde{\Phi}_h : C\left(a_{k+1}\right) = T_{a_{k+1}}\left(F\left(a_k\right)\right) \oplus L\left(a_{k+2}\right) \longrightarrow T_{a'_{k+1}}\left(F\left(a'_k\right)\right) \oplus L\left(a'_{k+2}\right).$$

Remark 6. *Any other prolongation of Φ, say $\widetilde{\Phi}_1$, differs from (2.1) by a linear map $\sigma_\Phi : L\left(a_{k+2}\right) \to T_{a'_{k+1}}\left(F\left(a'_k\right)\right)$, i.e. due to the isomorphisms above by*

$$\sigma_\Phi : \tau_a \to S^{k+1}\tau^*_{a'} \otimes \nu_{a'_0} \text{ or equivalently } \sigma_\Phi \in \tau^*_a \otimes S^{k+1}\tau^*_{a'} \otimes \nu_{a'_0}.$$

Making the identification $\Phi_h : \tau_a \simeq \tau_{a'}$ we see that $\widetilde{\Phi}_1$ preserves the metasymplectic structure if and only if $\delta\sigma_\Phi = 0$.

Proposition 4. *The following diagram*

$$\begin{tikzcd} S^{k+1}\tau^*_a \otimes \nu_{a_0} \arrow[r, "\widetilde{\Phi}_v"] \arrow[d, "\delta"'] & S^{k+1}\tau^*_{a'} \otimes \nu_{a'_0} \arrow[d, "\delta"] \\ \tau^*_a \otimes S^k\tau^*_a \otimes \nu_{a_0} \arrow[r, "\left(\widetilde{\Phi}^*_h\right)^{-1} \otimes \Phi_v"] & \tau^*_{a'} \otimes S^k\tau^*_{a'} \otimes \nu_{a'_0} \end{tikzcd}$$

commutes.

Proof. Indeed, from the constructions it follows that

$$\Phi_v\left(X\rfloor\delta\theta\right) = \widetilde{\Phi}_h\left(X\right)\rfloor\delta\widetilde{\Phi}_v\left(\theta\right) \tag{2.2}$$

for all $X \in \tau_a$ and $\theta \in S^{k+1}\tau^*_a \otimes \nu_{a_0}$. □

Theorem 5. *Mapping (2.1) preserves the metasymplectic structures.*

Proof. Let $X \in L\left(a_{k+2}\right) \simeq \tau_a$ and $\theta \in T_{a_{k+1}}\left(F\left(a_k\right)\right) = S^{k+1}\tau^*_a \otimes \nu_{a_0}$ be a horizontal and a vertical vectors. Their images are: $\widetilde{\Phi}_h\left(X\right) \in \tau_{a'}$ and $\widetilde{\Phi}_v\left(\theta\right) \in S^{k+1}\tau^*_{a'} \otimes \nu_{a'_0}$. Since Φ is a special metasymplectic transformation (for the exceptional case the arguments are similar) it maps vertical subspaces to the vertical ones: $S^k\tau^*_a \otimes \nu_{a_0} \to S^k\tau^*_{a'} \otimes \nu_{a'_0}$. Thus we get the mapping

$$S^k\tau_a \otimes \nu^*_{a_0} \ni \lambda \xrightarrow{\left(\Phi^*\right)^{-1}} \lambda' \in S^k\tau_{a'} \otimes \nu^*_{a'_0}.$$

With this fixed we have:

$$\begin{aligned} &\Omega_{\lambda'}\left(\widetilde{\Phi}_h\left(X\right), \widetilde{\Phi}^v\left(\theta\right)\right) = \left\langle \lambda', \widetilde{\Phi}_h\left(X\right)\rfloor\delta\widetilde{\Phi}^v\left(\theta\right)\right\rangle = \\ &\left\langle \lambda', \Phi^v\left(X\rfloor\delta\theta\right)\right\rangle = \left\langle \left(\Phi^v\right)^*\left(\lambda'\right), X\rfloor\delta\theta\right\rangle = \\ &\left\langle \lambda, X\rfloor\delta\theta\right\rangle = \Omega_\lambda\left(X, \theta\right). \end{aligned}$$

□

3. Framed jets and lifting of k-jets of bundle isomorphisms

3.1. Jets of bundle isomorphisms. Let $\phi : E \to E'$ be a local bundle isomorphism. Then the k-jet $[\phi]_a^k$ of ϕ at the point $a \in M$ determines a map

$$[\phi]_a^k : J_a^k \pi \to J_{\overline{\phi}(a)}^k \pi'$$

by the rule

$$[\phi]_a^k : [s]_a^k \longmapsto \left[\phi \circ s \circ \overline{\phi}^{-1}\right]_{\overline{\phi}(a)}^k$$

and the following diagram is obviously commutative

$$\begin{tikzcd} J_a^k \pi \arrow[r, "{[\phi]_a^k}"] \arrow[d, "\pi_{k,k-1}"'] & J_{\overline{\phi}(a)}^k \pi' \arrow[d, "\pi'_{k,k-1}"] \\ J_a^{k-1} \pi \arrow[r, "{[\phi]_a^{k-1}}"] & J_{\overline{\phi}(a)}^{k-1} \pi'. \end{tikzcd}$$

Moreover $[\phi]_a^k$ is an affine isomorphism when $k \geq 1$, and its linear part is equal to $S^k \left(\overline{\phi}_a{}^*\right)^{-1} \otimes \phi_{a_0}^v$.

These two properties completely characterize the k-jets.

Lemma 3. *Let $F : J_a^{k+1}\pi \to J_{a'}^{k+1}\pi'$ be a map such that*

1. *The diagram*

$$\begin{tikzcd} J_a^{k+1} \pi \arrow[r, "F"] \arrow[d, "\pi_{k+1,k}"'] & J_{a'}^{k+1} \pi' \arrow[d, "\pi'_{k+1,k}"] \\ J_a^{k} \pi \arrow[r, "{[\phi]_a^k}"] & J_{a'}^{k} \pi' \end{tikzcd}$$

   commutes for some bundle isomorphism ϕ. Here $a' = \overline{\phi}(a)$.
2. *F is an affine map and the linear part of the map at a point $a_k \in J^k\pi$ is equal to $S^{k+1}(A^*)^{-1} \otimes B : S^{k+1}\tau_a^* \otimes \nu_{a_0} \to S^{k+1}\tau_{a'}^* \otimes \nu_{\phi(a_0)}$ for some linear isomorphisms $A : \tau_a \to \tau_{a'}$ and $B : \nu_{a_0} \to \nu_{a_0'}$.*

Then there is a local bundle isomorphism ψ such that $[\psi]_a^{k+1} = F$ and $[\psi]_a^k = [\phi]_a^k$.

Proof. Let us consider

$$\Psi = \left([\phi]_a^{k+1}\right)^{-1} \circ F : J_a^{k+1}\pi \to J_a^{k+1}\pi.$$

Then Ψ induces the trivial transformation on $J_a^k\pi$ and an affine isomorphism on the bundle $\pi_{k+1,k} : J_a^{k+1}\pi \to J_a^k\pi$. Moreover linear part of this isomorphism is equal to $S^{k+1}\left(\widetilde{A}^*\right)^{-1} \otimes \widetilde{B}$ where $\widetilde{A} = \left(\overline{\phi}_{*,a}\right)^{-1} \circ A : \tau_a \to \tau_a$ and $\widetilde{B} = \left(\phi_{a_0}^v\right)^{-1} \circ B : \nu_{a_0} \to \nu_{a_0}$. Take any local bundle isomorphism λ such that $[\lambda]_a^k = \mathrm{id}$ and linear part of $[\lambda]_a^{k+1}$ coincides with $S^{k+1}\left(\widetilde{A}^*\right)^{-1} \otimes \widetilde{B}$. Then $\left([\lambda]_a^{k+1}\right)^{-1} \circ \Psi$ is a translation of the affine bundle and therefore can be represented by a smooth map $\upsilon : \pi^{-1}(a) \ni a_0 \longmapsto \upsilon(a_0) \in S^{k+1}\tau_a^* \otimes \nu_{a_0}$. Let us choose a π-vertical vector field X_υ on E such that X_υ has $(k+1)$-order zero on $\pi^{-1}(a)$ and $(k+1)$-jet of X_υ is equal to υ. Let C be a shift along X_υ on time $t = 1$. Then $(k+1)$-jet of C produces the translation in the affine bundle. In other words $[C]_a^{k+1} = \left([\lambda]_a^{k+1}\right)^{-1} \circ \Psi$ and $F = [\phi]_a^{k+1} \circ [\lambda]_a^{k+1} \circ [C]_a^{k+1}$. □

3.2. Framed jets.

Definition 9. *A pair* $\left([\phi]_a^k, \Phi\right)$, *where* $[\phi]_a^k : J_a^k\pi \to J_{a'}^k\pi'$ *is a* k*-jet of local bundle isomorphism and* $\Phi : a_k \in J_a^k\pi \longmapsto \Phi_{a_k}$ *is a smooth family of special metasymplectic transformations* $\Phi_{a_k} : C(a_k) \to C(a_k')$, $a_k' = [\phi]_a^k(a_k)$, *is called* framed k-jet.

Any framed k-jet $\left([\phi]_a^k, \Phi\right)$ determines a map $F : J_a^{k+1}\pi \to J_{a'}^{k+1}\pi'$, which is uniquely defined by the formula $\Phi_{a_k}(L(a_{k+1})) = L(F(a_{k+1}))$. This map satisfies the conditions of lemma 3 due to theorem 3.

Therefore, there is a local bundle isomorphism ψ such that $[\psi]_a^{k+1} = F$.

On the other hand, let us choose a section $H : J_a^{k+1}\pi \to J_a^{k+2}\pi$ of the projection $\pi_{k+2,k+1} : J_a^{k+2}\pi \to J_a^{k+1}\pi$. Then the construction of sec.2.4 shows that there is a family of metasymplectic transformations $\Phi_H : C(a_{k+1}) \to C\left([\psi]_a^{k+1}(a_{k+1})\right)$.

Summing up we get the following.

Proposition 5. *For any framed* k*-jet* $\left([\phi]_a^k, \Phi\right)$ *there exists a framed* $(k+1)$*-jet* $\left([\psi]_a^{k+1}, \Psi\right)$ *such that* $[\psi]_a^k = [\phi]_a^k$, $\Phi(L(a_{k+1})) = L\left([\psi]_a^{k+1}(a_{k+1})\right)$ *for all* $a_{k+1} \in J_a^{k+1}\pi$. *Moreover,* Φ *and* Ψ *have the same horizontal parts and vertical parts* Φ_v *and* Ψ_v *are conjugated via the isomorphisms* $\nu_{a_0} \to \nu_{a_0'}$ *and the projections* $\pi_{k+1,k}$, $\pi'_{k+1,k}$.

Definition 10. *We call the framed* $(k+1)$*-jet* $\left([\psi]_a^{k+1}, \Psi\right)$ 1*-st prolongation of the* k*-jet* $\left([\phi]_a^k, \Phi\right)$.

3.3. Prolongations of equivalences.

Definition 11. *A framed* k*-jet* $\left([\phi]_a^k, \Phi\right)$ *is called* framed isomorphism *of differential equations* $\mathcal{E} \subset J^k\pi$ *and* $\mathcal{E}' \subset J^k\pi'$ *if* $a_k' = [\phi]_a^k(a_k) \in \mathcal{E}'$, *and* $\Phi_{a_k}(C_{\mathcal{E}}(a_k)) = C_{\mathcal{E}'}(a_k')$ *for all* $a_k \in \mathcal{E}_a$.

Definition 12. *We call differential equation* $\mathcal{E} \subset J^k\pi$ 2*-solvable if the prolongations* $\mathcal{E}^{(1)} \subset J^{k+1}\pi$ *and* $\mathcal{E}^{(2)} \subset J^{k+2}\pi$ *are smooth manifolds and* $\pi_{k+2,k+1} : \mathcal{E}^{(2)} \to \mathcal{E}^{(1)}$ *and* $\pi_{k+1,k} : \mathcal{E}^{(1)} \to \mathcal{E}$ *are smooth bundles.*

Theorem 6. *Let differential equations* $\mathcal{E} \subset J^k\pi$ *and* $\mathcal{E}' \subset J^k\pi'$ *be* 2*-solvable. Assume that* $\left([\phi]_a^k, \Phi\right)$ *is a framed isomorphism of the differential equations. Then there exists a* 1*-st prolongation* $\left([\psi]_a^{k+1}, \Psi\right)$ *of* $\left([\phi]_a^k, \Phi\right)$ *which establishes a framed isomorphism of prolongations* $\mathcal{E}^{(1)} \subset J^{k+1}\pi$ *and* $\mathcal{E}'^{(1)} \subset J^{k+1}\pi'$.

Proof. The fact that Ψ_v establishes the isomorphism of prolongations of symbols follows from the naturality of the Spencer operator. The rest of the proof goes along the constructions in proposition 5. $\square$

The next result is the straightforward corollary of the above theorem.

Theorem 7. *Let differential equations $\mathcal{E} \subset J^k\pi$ and $\mathcal{E}' \subset J^k\pi'$ be formally integrable. Then any framed isomorphism $\left([\phi]_a^k, \Phi\right)$ can be lifted to framed isomorphisms of l-prolongations of the equations.*

4. Weyl tensors and one step lifting for non integrable PDEs

Now we consider the problem of formal equivalence for equations which are not formally integrable.

4.1. Weyl tensors. Consider differential equations $\mathcal{E} \subset J^k\pi$ and $\mathcal{E}' \subset J^k\pi'$. In this case we have the following decomposition of the Cartan space:

$$C_{\mathcal{E}}(a_k) = C(a_k) \cap T_{a_k}\mathcal{E} = g(a_k) \oplus H(a_k),$$

where $g(a_k) = T_{a_k}\mathcal{E} \cap T_{a_k}F(a_{k-1})$ is the symbol of the equation $\mathcal{E}$ at the point a_k and $H(a_k)$ is some horizontal space.

Note that horizontal subspace can be chosen isotropic $H(a_k) = L(a_{k+1})$ with a_{k+1} lying in the first prolongation $\mathcal{E}^{(1)}$ of the equation, of course in the case of non empty prolongation $\mathcal{E}^{(1)}$ over a_k.

We will identify $H(a_k) \simeq \tau_a$.

Let $\Omega_H = \Omega|_H$ be a "curvature" of $H = H(a_k)$.

Lemma 4. *Let H' be another horizontal space and $\sigma = \sigma_{H,H'} \in \tau_a^* \otimes g(a_k)$ be the corresponding operator. Then*

$$\Omega_{H'} = \Omega_H + \delta\sigma.$$

Proof. Let $\Omega_{H,\lambda} = \Omega_\lambda|_H = \Omega(\lambda)|_H$ for all $\lambda \in S^{k-1}\tau_a \otimes \nu_{a_0}^*$.
Then

$$\begin{aligned}
\Omega_{H',\lambda}(X_1, X_2) &= \Omega_\lambda(X_1 + \sigma(X_1), X_2 + \sigma(X_2)) = \\
&= \Omega_{H,\lambda}(X_1, X_2) + \Omega_\lambda(X_1, \sigma(X_2)) + \Omega_\lambda(\sigma(X_1), X_2) \\
&= \Omega_{H,\lambda}(X_1, X_2) + \langle \lambda, X_1 \rfloor \delta(\sigma(X_2)) - X_2 \rfloor \delta(\sigma(X_1)) \rangle \\
&= \Omega_{H,\lambda}(X_1, X_2) + \langle \lambda, \delta\sigma(X_1, X_2) \rangle .
\end{aligned}$$

$\square$

Since $\Omega_H \in \Lambda^2(\tau_a^*) \otimes S^{k-1}\tau_a^* \otimes \nu_{a_0}$ is δ-closed (generalized Bianchi identity, see [**11**]) we are led to

Definition 13. *The* Weyl tensor *of differential equation $\mathcal{E} \subset J^k\pi$ at point $a_k \in \mathcal{E}$ is the δ-cohomology class*

$$W_k(a_k) = \Omega_H \operatorname{mod} \delta(\tau_a^* \otimes g(a_k)) \in H^{k-1,2}(\mathcal{E}, a_k).$$

Theorem 8. *A Cartan connection at point $a_k \in \mathcal{E}$ can be chosen Bott if and only if $W(a_k) = 0$.*

REMARK 7. *This result can be reformulated as a criterion for* 1*-solvability of the equation at the point. Moreover the formal integrability criterion for the equation* $\mathcal{E}$ *can be formulated in terms of Weyl tensors*

$$W_k = 0, W_{k+1} = 0, \dots$$

together with the regularity condition that symbols of prolongations

$$g^{(l)} : \mathcal{E} \ni a_k \mapsto g^{(l)}(a_k) \subset S^{k+l}\tau_a^* \otimes \nu_{a_0}$$

are vector bundles (*see* [**11**]).

4.2. Lifting. Let $\mathcal{E} \subset J^k\pi$ be a differential equation such that 1-st prolongation $\mathcal{E}^{(1)} \subset J^{k+1}\pi$ exists but $\mathcal{E}^{(2)}$ does not (one can take Riemannian structure $\mathcal{E}_g$ on $J^1(M, \mathbb{R}^n)$ as an example).

We assume that $\pi_{k+1,k} : \mathcal{E}^{(1)} \to \mathcal{E}$ is a smooth bundle and a framed k-jet $\left([\phi]_a^k, \Phi\right)$ is an isomorphism between $\mathcal{E}$ and $\mathcal{E}'$. We shall denote by $A : \tau_a \to \tau_{a'}$ the horizontal and by $B : \nu_{a_0} \to \nu_{a_0'}$ the vertical parts of Φ.

Denote by $[\psi]_a^{k+1}$ the prolongation of ϕ given by Φ. We will try to define metasymplectic transformations $\Psi : C_{\mathcal{E}^{(1)}}(a_{k+1}) \to C_{\mathcal{E}'^{(1)}}\left(a'_{k+1}\right)$ by using a connection $H : a_{k+1} \longmapsto H(a_{k+1}) \subset C_{\mathcal{E}^{(1)}}(a_{k+1})$ on $\mathcal{E}^{(1)}$ and H' on $\mathcal{E}'^{(1)}$.

In other words, we define

$$\Psi_H : g^{(1)}(a_{k+1}) \oplus H(a_{k+1}) \to g^{(1)}\left(a'_{k+1}\right) \oplus H'\left(a'_{k+1}\right) \tag{4.1}$$

as the direct sum of $S^{k+1}(A^*)^{-1} \otimes B$ and A.

Note, that if Ψ_H is metasymplectic then it maps the Weyl tensor of $\mathcal{E}'^{(1)}$ to the corresponding one of $\mathcal{E}^{(1)}$.

REMARK 8. *It is easy to see that the condition "*Ψ_H *preserves the Weyl tensor" does not depend on a choice of the connection* H. *So this conditions depends on* Φ *only.*

THEOREM 9. *Let* Ψ_H *preserve the Weyl tensors. Then there is a Cartan connection* H *such that* Ψ_H *is a metasymplectic transformation.*

PROOF. Let H and $\widetilde{H}$ be Cartan connections on $\mathcal{E}^{(1)}$ and let σ be the indeterminacy element, $\sigma_{a_{k+1}} \in \tau_a^* \otimes g^{(1)}(a_{k+1})$. Denote by $q_\sigma : C(a_{k+1}) \to C(a_{k+1})$ the following linear operator: $q_\sigma(\theta) = \theta$ for vertical vectors and

$$q_\sigma(X) = X + X \rfloor \sigma$$

for horizontal vectors $X \in H(a_{k+1})$.

Then $q_\sigma(H) = \widetilde{H}$ and

$$\Psi_{\widetilde{H}} = \Psi_H \circ q_\sigma^{-1}.$$

We have

$$\Psi_{\widetilde{H}}^*(\Omega'_H) = \left(q_\sigma^{-1}\right)^*(\Psi_H^*\Omega'_H).$$

Assume that Φ_H preserves the Weyl tensors. Then

$$\Psi_H^*(\Omega'_H) = \Omega_H + \delta\theta$$

on $C_{\mathcal{E}^{(1)}}(a_{k+1})$ for some $\theta \in \tau_a^* \otimes g(a_k)$.

Therefore, $\Psi_{\widetilde{H}}^*(\Omega'_H) = \Omega_H$ if and only if $(q_\sigma)^*(\Omega) = \Omega + \delta\theta$ for some σ. The following lemma shows then we can take $\sigma = \theta$. □

Lemma 5. *$(q_\sigma)^* (\Omega_H) = \Omega_H + \delta\sigma$.*

Proof. Both forms coincide on pairs of vectors (X,Y) if at least one vector is vertical. Take $X, Y \in H(a_{k+1})$.

We get

$$(q_\sigma)^* (\Omega_H)(X,Y) = \Omega(X + X\rfloor\sigma, Y + Y\rfloor\sigma) = \Omega(X,Y) + \Omega(X, Y\rfloor\sigma) + \Omega(X\rfloor\sigma, Y) =$$
$$\Omega(X,Y) + X\rfloor\delta(Y\rfloor\sigma) - Y\rfloor\delta(X\rfloor\sigma) = \Omega_H(X,Y) + \delta\sigma(X,Y).$$

□

4.3. Tangency and equivalence.

Theorem 10. *Let $\mathcal{E}, \mathcal{E}' \subset J^k\pi$ be differential equations situated in the same jet space. Assume that $a_k \in \mathcal{E} \cap \mathcal{E}'$, and*

1. *Projections are tangent: $T_{a_{k-1}}(\pi_{k,k-1}\mathcal{E}) = T_{a_{k-1}}(\pi_{k,k-1}\mathcal{E}')$.*
2. *Fibres $F(a_k) \cap \mathcal{E}^{(1)}$ and $F(a_k) \cap \mathcal{E}'^{(1)}$ coincide.*

Then the equations are tangent at $a_k : T_{a_k}(\mathcal{E}) = T_{a_{k-1}}(\mathcal{E}')$.

Proof. Cartan subspace $C_{\mathcal{E}}(a_k)$ is completely determined by non vertical vectors, which belong to $L(a_{k+1})$ for $a_{k+1} \in \mathcal{E}^{(1)}$. Hence Cartan subspaces are the same for both equations due to fibers coinciding. The complementary parts are tangent also because of the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & g(a_k) \arrow[r] & T_{a_k}(\mathcal{E}) \arrow[r, "(\pi_{k,k-1})_*"] & T_{a_{k-1}}(\pi_{k,k-1}\mathcal{E}) \arrow[r] & 0 \\
0 \arrow[r] & g(a_k) \arrow[r] \arrow[u] & C_{\mathcal{E}}(a_k) \arrow[r, "(\pi_{k,k-1})_*"] \arrow[u, hook] & L(a_k) \arrow[r] \arrow[u, hook] & 0.
\end{tikzcd}$$

Indeed from the diagram we get

$$T_{a_k}(\mathcal{E}) \diagup C_{\mathcal{E}}(a_k) \simeq T_{a_{k-1}}(\pi_{k,k-1}\mathcal{E}) \diagup L(a_k) \simeq T_{a_k}(\mathcal{E}') \diagup C_{\mathcal{E}'}(a_k).$$

□

In the same way we get the similar result.

Theorem 11. *Let $\mathcal{E}, \mathcal{E}' \subset J^k\pi$ be differential equations situated in the same jet space. Assume that $a_k \in \mathcal{E} \cap \mathcal{E}'$ and*

1. *Projections are tangent: $T_{a_{k-1}}(\pi_{k,k-1}\mathcal{E}) = T_{a_{k-1}}(\pi_{k,k-1}\mathcal{E}')$.*
2. *Cartan subspaces $C_{\mathcal{E}}(a_k)$ and $C_{\mathcal{E}'}(a_k)$ coincide.*

Then the equations are tangent at $a_k : T_{a_k}(\mathcal{E}) = T_{a_k}(\mathcal{E}')$.

Let us apply these results to the situation described in the previous section.

Theorem 12.

1. *Let framed k-jet $\left([\phi]_a^k, \Phi\right)$ be an isomorphism of $\mathcal{E}$ and $\mathcal{E}'$ and let $[\psi]_a^{k+1}$ be the lifting of $[\phi]_a^k$ determined by Φ. Then differential equations $\psi^{(k)}(\mathcal{E})$ and $\mathcal{E}'$ are tangent at any point $a'_k \in \mathcal{E}' \cap J^k_{a'}\pi'$.*
2. *If the map Ψ_H constructed by Φ preserves the Weyl tensors then $\widetilde{\psi}^{(k+1)}\left(\mathcal{E}^{(1)}\right)$ and $\mathcal{E}'^{(1)}$ are tangent at any point $a'_{k+1} \in \mathcal{E}'^{(1)} \cap J^{k+1}_{a'}\pi'$ for some bundle isomorphism $\widetilde{\psi}$ where $\left[\widetilde{\psi}\right]_a^{k+1} = [\psi]_a^{k+1}$.*

References

[1] D. Alekseevskij, V. Lychagin, A. Vinogradov, *"Basic ideas and concepts of differential geometry"*, *Encyclopedia of mathematical sciences*, **28**, Geometry 1, Springer-Verlag (1991)

[2] E. Cartan, *"Euvres completes"*, Paris, Gautier-Villars (1953)

[3] H. Goldschmidt, *"Integrability criteria for systems of nonlinear partial differential equations"*, J. Diff. Geom., **1**(3) (1967), 269–307

[4] I. Krasilschik, V. Lychagin, A. Vinogradov, *"Geometry of jet spaces and differential equations"*, Gordon and Breach (1986)

[5] B. S. Kruglikov, *"Symplectic and contact Lie algebras with an application to Monge-Ampère equations"*, Proc. of Steklov Mathematical Inst., **221**, 232–246 (1998)

[6] B. S. Kruglikov, *"Classification of Monge-Ampère equations with two variables"*, Banach Center Publications, Institute of Math., PAS, Warszawa (1998)

[7] A. Kumpera, D. Spencer, *"Lie equations. Volume 1: General theory"*, Princeton University Press and University Tokyo Press (1972)

[8] S. Lie, *"Gessamelte Abhandlungen"*, Band VI, Aschehoug, Oslo and Teubner, Leipzig (1922)

[9] V. V. Lychagin, *"Geometric theory of singularities of solutions of nonlinear differential equations"*, J. Soviet Math., **51** (1990), 2735–2757

[10] V. V. Lychagin, *"Homogeneous structures on manifolds: differential geometry from PDEs viewpoint"*, Mathematical Notes, **51**, no. 4 (1992), 54–68

[11] V. V. Lychagin, *"Homogeneous geometric structures and homogeneous differential equations"*, in *Amer. Math. Soc. Transl.*, "The interplay between differential geometry and differential equations", V. Lychagin Eds., ser. 2, **167** (1995), 143–164

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