Point classification of 2nd order ODEs: Tresse classification revisited and beyond

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Summary. In 1896 Tresse gave a complete description of relative differential invariants for the pseudogroup action of point transformations on the 2nd order ODEs. The purpose of this paper is to review, in light of modern geometric approach to PDEs, this classification and also discuss the role of absolute invariants and the equivalence problem.[1]

Introduction

Second order scalar ordinary differential equations have been the classical target of investigations and source of inspiration for complicated physical models. Under contact transformations all these equations are locally equivalent, but to find such a transformation for a pair of ODEs is the same hard problem as to find a general solution, which as we know from Ricatti equations is not always possible.

Most integration methods for second order ODEs are related to another pseudogroup action – point transformations, which do not act transitively on the space of all such equations. All linear 2nd order ODEs are point equivalent.

S. Lie noticed that ODEs linearizable via point transformations have necessarily cubic nonlinearity in the first derivatives and described a general test to construct this linearization map [Lie_2]. Later R. Liouville found precise conditions for linearization [Lio]. But it was A. Tresse who first wrote the complete set of differential invariants for general 2nd order ODEs.

The paper [Tr_2] is a milestone in the geometric theory of differential equations, but mostly one result (linearization of S.Lie-R.Liouville-A.Tresse) from the manuscript is used nowadays. In this note we would like to revise the Tresse classification in modern terminology and provide some alternative for-

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mulations and proofs. We make relation to the equivalence problem more precise and also compare this approach with E.Cartan's equivalence method.

This classification can illustrate the finite representation theorem for differential invariants algebra, also known as Lie-Tresse theorem. The latter in the ascending degree of generality was proven in different sources [Lie_1, Tr_1, Ov, Ku, Ol, KL_1]. In particular, the latter reference contains the full generality statement, when the pseudogroup acts on a system of differential equations $\mathcal{E} \subset J^l(\pi)$ (under regularity assumption, see also [SS]). We refer to it for details and further references and we also cite [KLV, KL_2] as a source of basic notations, methods and results.

The structure of the paper is the following. In the first section we provide a short introduction to scalar differential invariants of a pseudogroup action and recall what the algebra of relative differential invariants is. In Section 2 we review the results of Tresse, confirming his formulae with independent computer calculation. In Section 3 we complete Tresse's paper by describing the algebra of absolute invariants and proving the equivalence theorem (in [Tr_2] this was formulated via relative invariants, which makes unnecessary complications with homogeneity, and only necessity of the criterion was explained). In Section 4 we discuss the non-generic 2nd order equations, which contain in particular linearizable ODEs. Section 5 is devoted to discussion of symmetric ODEs.

Finally in Appendix (written jointly with V.Lychagin) we provide another approach to the equivalence problem, based on a reduction of an infinite-dimensional pseudogroup action to a Lie group action.

1 Scalar differential invariants

We refer to the basics of pseudogroup actions to [Ku, KL_2], but recall the relevant theory about differential invariants (see also [Tr_1, Ol]). We'll be concerned with the infinite Lie pseudogroup $G = \text{Diff}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ with the corresponding Lie algebras sheaf (LAS) $\mathfrak{g} = \mathfrak{D}_{\text{loc}}(\mathbb{R}^2)$ of vector fields.

The action of G has the natural lift to an action on the space $J^\infty\pi$ for an appropriate[2] vector bundle π, provided we specify a Lie algebras homomorphism $\mathfrak{g} \to \mathfrak{D}_{\text{loc}}(J^0\pi)$. Then we can restrict to the action of formal LAS $J^\infty(\mathbb{R}^2, \mathbb{R}^2)$.

A function $I \in C^\infty(J^\infty\pi)$ (this means that I is a function on a finite jet space $J^k\pi$ for some $k > 1$) is called a (scalar absolute) differential invariant if it is constant along the orbits of the lift of the action of G to $J^k\pi$.

For connected groups G we have an equivalent formulation: I is an (absolute) differential invariant if the Lie derivative vanishes $L_{\hat{X}}(I) = 0$ for all vector fields X from the lifted action of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$.

[2] In this paper $\pi = M \times \mathbb{R}$ is a trivial 1-dimensional bundle over $M \simeq \mathbb{R}^3$, so $J^k\pi = J^k M$.

Note that often functions I are defined only locally near families of orbits. Alternatively we should allow I to have meromorphic behavior over smooth functions (but we'll be writing though about local functions in what follows, which is a kind of micro-locality, i.e. locality in finite jet-spaces).

The space $\mathcal{I} = \{I\}$ forms an algebra with respect to usual algebraic operations of linear combinations over $\mathbb{R}$ and multiplication and also the composition $I_1, \dots, I_s \mapsto I = F(I_1, \dots, I_s)$ for any $F \in C^\infty_{\text{loc}}(\mathbb{R}^s, \mathbb{R})$, $s = 1, 2, \dots$ any finite number. However even with these operations the algebra $\mathcal{I}$ is usually not locally finitely generated. Indeed, the subalgebras $\mathcal{I}_k \subset \mathcal{I}$ of order k differential invariants are finitely generated on non-singular strata with respect to the above operations, but their injective limit $\mathcal{I}$ is not.

However finite-dimensionality is restored if we add invariant derivatives, i.e. $\mathscr{C}$-vector fields $\vartheta \in C^\infty(J^\infty\pi) \otimes_{C^\infty(M)} \mathfrak{D}(M)$ commuting with the G-action on the bundle π. These operators map differential invariants to differential invariants $\vartheta : \mathcal{I}_k \to \mathcal{I}_{k+1}$.

Lie-Tresse theorem claims that the algebra of differential invariants $\mathcal{I}$ is finitely generated with respect to algebraic-functional operations and invariant derivatives.

A helpful tool on the practical way to calculate algebra $\mathcal{I}$ of invariants are relative invariants, because they often occur on the lower jet-level than absolute invariants. A function $F \in C^\infty(J^\infty\pi)$ is called a relative scalar differential invariant if the action of pseudogroup G writes

$$g^* F = \mu(g) \cdot F$$

for a certain weight, which is a smooth function $\mu : G \to C^\infty(J^\infty\pi)$, satisfying the axioms of multiplier representation

$$\mu(g \cdot h) = h^* \mu(g) \cdot \mu(h), \quad \mu(e) = 1.$$

The corresponding infinitesimal analog for an action of LAS $\mathfrak{g}$ is given via a smooth map (the multiplier representation is denoted by the same letter) $\mu : \mathfrak{g} \to \mathfrak{D}(J^\infty\pi)$, which satisfies the relations

$$\mu_{[X,Y]} = L_{\hat{X}}(\mu_Y) - L_{\hat{Y}}(\mu_X), \quad \forall\, X, Y \in \mathfrak{g},$$

Then a relative scalar invariant is a function $F \in C^\infty(J^\infty\pi)$ such that $L_{\hat{X}} I = \mu_X \cdot I$. In other words (in both cases) the equation $F = 0$ is invariant under the action.

Let $\mathfrak{M} = \{\mu_X\}$ be the space of admissible weights[3]. Denote by $\mathcal{R}^\mu$ the space of scalar relative differential invariants of weight μ. Then

$$\mathcal{R} = \bigcup_{\mu \in \mathfrak{M}} \mathcal{R}^\mu$$

[3] It is given via a certain cohomology theory, which will be considered elsewhere.

is a $\mathfrak{M}$-graded module over the algebra of absolute scalar differential invariants $\mathcal{I} = \mathcal{R}^0$ corresponding to the weight $\mu = 0$ for the LAS action ($\mu = 1$ for the pseudogroup action).

The space $\mathfrak{M}$ of weights (multipliers) is always a group, but we can transform it into a $\mathbf{k}$-vector space ($\mathbf{k} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) by taking tensor product $\mathfrak{M}\otimes\mathbf{k}$ and considering (sometimes formal) combinations $(I_1)^{\alpha_1}\cdots(I_s)^{\alpha_s}$. Then we have:

$$\mathcal{R}^{\mu}\cdot\mathcal{R}^{\bar{\mu}} \subset \mathcal{R}^{\mu+\bar{\mu}}, \quad (\mathcal{R}^{\mu})^{\alpha} \subset \mathcal{R}^{\alpha\cdot\mu}.$$

2 Tresse classification revisited

We start by re-phrasing the main results of Tresse classification[4].

2.1 Relative differential invariants of 2nd order ODEs

The point transformation LAS $\mathfrak{D}_{\rm loc}(J^0\mathbb{R})$, with $J^0\mathbb{R}(x) = \mathbb{R}^2(x,y)$, equals $\mathfrak{g} = \{\xi_0 = a\partial_x + b\partial_y : a = a(x,y), b = b(x,y)\}$ and it prolongs to the subalgebra

$$\mathfrak{g}_2 = \{\xi = a\partial_x + b\partial_y + A\partial_p + B\partial_u\} \subset \mathfrak{D}_{\rm loc}(J^2\mathbb{R}), \quad J^2\mathbb{R} = \mathbb{R}^4(x,y,p,u),$$
$$A = b_x - (a_x - b_y)p - a_y p^2,\ B = B_0 + uB_1,$$
$$B_0 = b_{xx} - (a_x - 2b_y)_x p - (2a_x - b_y)_y p^2 - a_{yy}p^3,\ B_1 = -(2a_x - b_y) - 3a_y p$$

where we denote $p = y'$, $u = y''$ the jet coordinates.

Using the notations $D_x = \partial_x + p\,\partial_y$, $\varphi = (dy - p\,dx)(a\partial_x + b\partial_y) = b - p\,a$ (we'll see soon these show up naturally), these expressions can be rewritten as

$$A = D_x(\varphi),\ B_0 = D_x^2(\varphi),\ B_1 = \partial_y(\varphi) - 2D_x(a)$$

Thus the LAS $\mathfrak{h} = \mathfrak{g}_2 \subset \mathfrak{D}_{\rm loc}(J^0\mathbb{R}^3(x,y,p))$ being given we represent a second order ODE as a surface $u = f(x,y,p)$ in $J^0\mathbb{R}^3(x,y,p) = \mathbb{R}^4(x,y,p,u)$ and $k^{\rm th}$ order differential invariants of this ODE are invariant functions $I \in C^\infty_{\rm loc}(J^k\mathbb{R}^3)$ of the prolongation

$$\mathfrak{h}_k = \{\hat{\xi} = a\,\mathcal{D}_x + b\,\mathcal{D}_y + A\,\mathcal{D}_p + \sum_{|\sigma|\le k}\mathcal{D}_\sigma^{(k)}(f)\,\partial_{u_\sigma}\} \subset \mathfrak{D}(J^k\mathbb{R}^3),$$
$$f = B_0 + B_1 u - a\,u_x - b\,u_y - A\,u_p \ : \qquad \hat{\xi}(I) = 0.$$

Here $\mathcal{D}_\sigma^{(k)} = \mathcal{D}_\sigma|_{J^k}$ with $\mathcal{D}_\sigma = \mathcal{D}_x^l\mathcal{D}_y^m\mathcal{D}_p^n$ for $\sigma = (l\cdot 1_x + m\cdot 1_y + n\cdot 1_p)$, so that

[4] We use different notations p instead of z, u instead of ω etc, but this is not crucial.

$$\mathcal{D}_\sigma(f) = \mathcal{D}_\sigma(B_0) + \sum \frac{|\tau|!}{\tau!}\Big(\mathcal{D}_\tau(B_1)u_{\sigma-\tau} - D_\tau(a)u_{\sigma-\tau+1_x} - \mathcal{D}_\tau(b)u_{\sigma-\tau+1_y} - \mathcal{D}_\tau(A)u_{\sigma-\tau+1_p}\Big).$$

In the above formula we used the usual partial derivatives ∂_x etc in the total derivative operators $\mathcal{D}_\sigma$ etc. All these operators commute.

It is more convenient, following Tresse, to use the operator $D_x = \partial_x + p\,\partial_y$ on the base instead and to form the corresponding total derivative $\hat{\mathcal{D}}_x = \mathcal{D}_x + p\mathcal{D}_y$. These operators will no longer commute and we need a better notation for the corresponding non-holonomic partial derivatives.

Denote $u^k_{lm} = \hat{\mathcal{D}}^l_x \mathcal{D}^m_y \mathcal{D}^k_p(u)$, which equals u_{lmk} mod (lower order terms). The first relative invariants calculated by Tresse have order 4 and are:

$$I = u^4, \quad H = u^2_{20} - 4u^1_{11} + 6u_{02} + u(2u^3_{10} - 3u^2_{01}) - u^1(u^2_{10} - 4u^1_{01}) + u^3 u_{10} - 3u^2 u_{01} + u\cdot u\cdot u^4.$$

In this case the weights form two-dimensional lattice and the relative invariants are

$$\mathcal{R}^{r,s} = \{\psi \in C^\infty(J^\infty\mathbb{R}^3) : \hat{\xi}(\psi) = -(rD_x(a) + s\partial_y(\varphi))\psi\}$$

Note that $\hat{\xi}(\psi) = -(w\,C^w_\xi + q\,C^q_\xi)\psi$ for $w = r$, $q = s - r$ (weight and quality in Tresse terminology). Here the coefficients can be expressed as operators of $\xi_0 = a\partial_x + b\partial_y$ and $\xi_1 = a\partial_x + b\partial_y + A\partial_p$:

$$C^w_\xi = a_x + b_y = \mathrm{div}_{\omega_0}(\xi_0) \text{ and } C^q_\xi = \partial_y(\varphi) = \tfrac{1}{2}\,\mathrm{div}_{\Omega_0}(\xi_1)$$

with $\omega_0 = dx \wedge dy$ the volume form on $J^0\mathbb{R}$ and $\Omega_0 = -\omega \wedge d\omega$ on $J^1\mathbb{R}$, where $\omega = dy - p\,dx$ is the standard contact form of $J^1\mathbb{R}$. These two form the base of all weights[5].

There are relative invariant differentiations[6] (differential parameters in the classical language):

$$\Delta_p = \mathcal{D}_p + (r-s)\frac{u^5}{5u^4} : \mathcal{R}^{r,s} \to \mathcal{R}^{r-1,s+1},$$

$$\Delta_x = \hat{\mathcal{D}}_x + u\,\Delta_p + \Big((3r+2s)\Big(u^1 + \frac{3u\,u^5}{5u^4}\Big) + (2r+s)\frac{u^4_{10}}{u^4}\Big) : \mathcal{R}^{r,s} \to \mathcal{R}^{r+1,s},$$

$$\Delta_y = \mathcal{D}_y + \frac{u^5}{5u^4}\,\Delta_x + \Big(2\,u^1 + \frac{u^4_{10} + u\,u^5}{u^4}\Big)\Delta_p + \Big((r+2s)\frac{u^4_{01}}{4u^4} + (3r+2s)\Big(\frac{u^2}{8} + \frac{3}{20}\frac{u^5(u^4_{10} + u\,u^5 + 2u^1u^4)}{u^4u^4}\Big)\Big) : \mathcal{R}^{r,s} \to \mathcal{R}^{r,s+1}.$$

[5] This is a result from a joint discussion with V.Lychagin. It is important since in Tresse [Tr$_2$] this is an ad-hoc result, based on the straightforward calculations, but not fully justified. More details will appear in a separate publication.

[6] Note that they are differential operators of the 1st order, obtained from the base derivations via an invariant connection.

Theorem 1. [Tr$_2$] *The space of relative differential invariants $\mathcal{R}$ is generated by the invariant H and differentiations $\Delta_x, \Delta_y, \Delta_p$ on the generic stratum.*

Notice that the latter two 1st order $\mathscr{C}$-differential operators have the form:

$$\Delta_x = \mathcal{D}_x + p\,\mathcal{D}_y + u\,\mathcal{D}_p + r\Bigl(3u^1 + 2\frac{u\,u^5 + u^4_{10}}{u^4}\Bigr) + s\Bigl(2u^1 + \frac{u\,u^5 + u^4_{10}}{u^4}\Bigr),$$

$$\Delta_y = \frac{u^5}{5u^4}\,\mathcal{D}_x + \Bigl(1 + p\frac{u^5}{5u^4}\Bigr)\mathcal{D}_y + \Bigl(2\,u^1 + \frac{5u^4_{10} + 6u\,u^5}{u^4}\Bigr)\mathcal{D}_p + r\Bigl(\frac{3u^2}{8} + \frac{u^4_{01}}{4u^4} + \frac{19u^1u^5}{10u^4} + \frac{21(u\,u^5 + u^4_{01})u^5}{20\,u^4\cdot u^4}\Bigr) + s\Bigl(\frac{u^2}{4} + \frac{u^4_{01}}{2u^4} + \frac{3u^1u^5}{5u^4} + \frac{3(u\,u^5 + u^4_{01})u^5}{10\,u^4\cdot u^4}\Bigr),$$

and so $\Delta_x, \Delta_y, \Delta_p$ are linearly independent everywhere outside $I = 0$.

2.2 Specifications

Several remarks are noteworthy in relation with the theorem:

1. The number of basic relative differential invariants of pure order k is given in the following table

$k:$	0	1	2	3	4	5	6	7	8	...	k	...
$\#:$	0	0	0	0	2	3	11	17	24	...	$\frac{1}{2}(k^2 - k - 8)$	

The generators in order 4 are $I \in \mathcal{R}^{-2,3}$ and $H \in \mathcal{R}^{2,1}$; in order 5 $H_{10} = \Delta_x(H) \in \mathcal{R}^{3,1}$, $H_{01} = \Delta_y(H) \in \mathcal{R}^{2,2}$ and $K = \Delta_p(H) \in \mathcal{R}^{1,2}$; in order 6 are[7] $(H_{20}, H_{11}, H_{02}) \in \mathcal{R}^{4,1} \oplus \mathcal{R}^{3,2} \oplus \mathcal{R}^{2,3}$, $(K_{10}, K_{01}) \in \mathcal{R}^{2,2} \oplus \mathcal{R}^{1,3}$ and $\Omega^l_{ij} = u^l_{ij} +$(lower terms for certain order on monomials) $\in \mathcal{R}^{i+2-l,j+l-1}$, $\deg \Omega^l_{ij} = i + j + l = 6$, $l > 3$:

order k	basic relative differential invariants
4	$I,\ H$
5	$H_{10},\ H_{01},\ K$
6	$H_{20},\ H_{11},\ H_{02},\ K_{10},\ K_{01},\ \Omega^4_{20},\ \Omega^4_{11},\ \Omega^4_{02},\ \Omega^5_{10},\ \Omega^5_{01},\ \Omega^6$

Thus in ascending order k, we must add the generators I, H and then Ω^{6-i-j}_{ij}, $i + j \le 2$ (one encounters the relations $\Delta_x(I) = \Delta_y(I) = \Delta_p(I) = 0$). Invariants of order $k > 6$ are obtained via invariant derivations from the lower order.

2. The theorem as formulated gives only generators. The relations (differential syzygies) are the following (also contained in [Tr$_2$]):

[7] We let $H_{ij} = \Delta^i_x \Delta^j_y H$ and $K_{ij} = \Delta^i_x \Delta^j_y K$, though in [Tr$_2$] there is a difference between $\Delta_x K$ and K_{10}, $\Delta_y K$ and K_{01}. Since this only involves a linear transformation, this is possible.

$$[\Delta_p,\Delta_x]=\Delta_y+\frac{3(3r+2s)}{5}\frac{\Omega^5_{10}}{I}$$
$$[\Delta_p,\Delta_y]=\frac{\Omega^6}{5I}\Delta_x+\frac{\Omega^5_{10}}{I}\Delta_p-\frac{3(3r+2s)}{20}\frac{\Omega^5_{01}}{I}$$
$$[\Delta_x,\Delta_y]=\frac{\Omega^5_{10}}{5I}\Delta_x+\frac{\Omega^4_{20}}{I}\Delta_p-\frac{3(3r+2s)}{4}\frac{\Omega^4_{11}}{I}$$

together with the following relations for coefficients-invariants (the first of which is just the application of the above commutator relation)

$$\Omega^5_{10}=\frac{5I}{24H}([\Delta_p,\Delta_x]H-\Delta_yH),\qquad \Omega^5_{01}=\frac{4}{9}(\Delta_p\Omega^5_{10}-\Delta_x\Omega^6),$$
$$\Omega^4_{20}=\Delta_p^2H-\frac{\Omega^6}{5I}H,\qquad \Omega^4_{11}=\frac{4}{3}(\Delta_p\Omega^4_{20}-\Delta_x\Omega^5_{10}).$$

It is important that the relation for the last additional invariant of order 6

$$\Omega^4_{02}=\frac{4}{5}(\Delta_y\Omega^5_{10}-\Delta_x\Omega^5_{01}+\frac{5\Omega^4_{20}\Omega^6+\Omega^5_{10}\Omega^5_{01}}{5I})$$

can be considered as definition, while first additional invariant[8] of order 6

$$\Omega^6=u^6-\frac{6}{5}\frac{u^5\cdot u^5}{u^4}$$

can be obtained from a higher relation via application of the relation for $[\Delta_p,\Delta_y]$ to H and K.

Thus we see that involving syzygy of higher order invariants (prolongation-projection) we can restore the invariants I,Ω^k_{ij} from H and invariant differentiations Δ_j, as the theorem claims.

3. The theorem specifies the relative invariants only on the generic stratum. If we take the minimal number of generators $(H,\Delta_x,\Delta_y,\Delta_p)$, then this stratum is specified by a number of non-degeneracy conditions of high order.

However if we take more generators $(I,H,\Omega^6,\Delta_x,\Delta_y,\Delta_p)$, or the collection of basic invariants $(I,H,\Omega^6,\Omega^5_{10},\dots,\Omega^4_{02},\Delta_x,\Delta_y,\Delta_p)$ for the completeness in ascending order k, then this condition is very easy: just $I\neq 0$.

Notice that the condition $I=0$ is important, since it describes the singular stratum (see however §1.4.2 where this case is handled).

3 Classification of 2nd order ODEs

While a complete classification of relative differential invariants for 2nd order scalar ODEs was achieved by Tresse, absolute invariants are not described in [Tr$_2$]. They however can be easily deduced.

[8] This invariant is important with another approach, see Appendix.

3.1 Dimensional count

Let us at first count the number of absolute invariants on a generic stratum[9]. This number equals the codimension of a generic orbit in the corresponding jet-space.

Denote by $\mathcal{O}_k$ the orbit through a generic point in $J^k\mathbb{R}^3(x,y,p)$ of the pseudogroup of point transformations. Tangent to it is determined by the corresponding LAS and so we can calculate codimension of the orbit. Indeed, denoting by St_k the stabilizer of the LAS $\mathfrak{h}_k$ at the origin we get

$$\dim \mathcal{O}_k = \operatorname{codim} \mathrm{St}_k\,.$$

To calculate the stabilizer we should adjust the normal form of the equation at the origin via a point transformation. This can be done via a projective configuration (Desargues-type) theorem of [A] (§1.6): any 2nd order ODE $y'' = u(x,y,p)$, $p = y'$, can be transformed near a given point to

$$y'' = \alpha(x)y^2 + o(|y|^3 + |p|^3).$$

Denote by $\mathfrak{m}$ the maximal ideal at the given point (so $\mathfrak{m}^k$ is the space of functions vanishing to order k). Then we can suppose that at a given point

$$u, u_x, u_y, u_p, u_{xx}, u_{xy}, u_{xp}, u_{yp}, u_{pp} \in \mathfrak{m}.$$

Therefore the stabilizer St_k is given by the union of the following conditions on the coefficients of $\hat{\xi} \in \mathfrak{h}_k$ (equivalently on coefficients of $\xi_0 \in \mathfrak{g}$)

$$a \in \mathfrak{m}^{k-2},\ a_{yy} \in \mathfrak{m}^{k-3},\ b \in \mathfrak{m}^{k-1},\ b_{xx} \in \mathfrak{m}^{k},$$
$$a_x \in \mathfrak{m}^{k-2},\ (2a_x - b_y)_y \in \mathfrak{m}^{k-2},\ (a_x - 2b_y)_x \in \mathfrak{m}^{k-1}.$$

Thus the Taylor expansion of $a = a(x,y)$ can contain only the following monomials

$$\{x^iy^j : i+j \le k-1\},\ \{x^iy^{k-i} : i > 1\},\ \{x^iy^{k+1-i} : i > 2\}$$

and the allowed monomials for $b = b(x,y)$ are

$$\{x^iy^j : i+j \le k\},\ \{x^iy^{k+1-i} : i \ge 1\},\ \{x^iy^{k+2-i} : i \ge 2\}.$$

This yields that $\operatorname{codim}(\mathrm{St}_k)$ equals:

$$\dim\bigl(\mathbb{C}[x,y]^2/\,\mathrm{St}_k\bigr) = \frac{k(k+1)}{2} + 2(k-1) + \frac{(k+1)(k+2)}{2} + 2(k+1) = k^2 + 6k + 1$$

and so the number $\imath_k$ of the basic differential invariants of order $\le k$ is equal to

[9] This count is independent of Tresse argumentation, and so together with footnote[5] it provides a rigorous proof of the table in §1.2.2.

$$\text{\ae}_k = \operatorname{codim} \mathcal{O}_k = \dim J^k\mathbb{R}^3 - \dim \mathcal{O}_k$$
$$= 3 + \frac{(k+1)(k+2)(k+3)}{6} - (k^2+6k+1) = \frac{k^3-25k+18}{6}.$$

As this formula indicates for $k \le 4$ the generic orbit is open, so that such stratum has no absolute invariants (however for $k = 4$ there are singular orbits, so that the relative invariants I, H appear).

In order $k = 5$ the formula yields $\text{\ae}_5 = 3$ differential invariants. For $k > 5$ we deduce the number of pure order k basic differential invariants:

$$\text{\ae}_k - \text{\ae}_{k-1} = \frac{k(k-1)}{2} - 4.$$

3.2 Absolute differential invariants

There are two ways of adjusting a basis on the lattice $\mathfrak{M}$ of weights via relative invariants. As follows from specification for $\mathbb{Z}^2$-lattice of weights from §1.2.2, the basic invariants are

$$J_1 = I^{-1/8}H^{3/8} \in \mathcal{R}^{1,0}, \quad J_2 = I^{1/4}H^{1/4} \in \mathcal{R}^{0,1}.$$

Another choice, which allow to avoid branching but increase the order, is

$$\tilde{J}_1 = \frac{H_{10}}{H} \in \mathcal{R}^{1,0}, \quad \tilde{J}_2 = \frac{H_{01}}{H} \in \mathcal{R}^{0,1}.$$

Then (choosing J_i or $\tilde{J}_i$) we get isomorphism for $k > 4$:

$$\mathcal{R}^{r,s}_k/\mathcal{R}^{r,s}_{k-1} \simeq \mathcal{I}_k/\mathcal{I}_{k-1}, \qquad F \mapsto F/(J_1^r J_2^s).$$

Thus with any choice the list of basic differential invariants in order 5 is

$$\bar{H}_{10} = H_{10}/(J_1^3 J_2),\ \bar{H}_{01} = H_{01}/(J_1^2 J_2^2),\ \bar{K} = K/(J_1 J_2^2)$$

and in pure order 6 is

$$\bar{H}_{20} = H_{20}/(J_1^4 J_2),\ \bar{H}_{11} = H_{11}/(J_1^3 J_2^2),\ \bar{H}_{02} = H_{02}/(J_1^2 J_2^3),\ \bar{K}_{10} = K_{10}/(J_1^2 J_2^2),$$
$$\bar{K}_{01} = K_{01}/(J_1 J_2^3),\ \bar{\varOmega}^4_{20} = \varOmega^4_{20}/(J_2^3),\ \bar{\varOmega}^4_{11} = \varOmega^4_{11}/(J_1^{-1} J_2^4),\ \bar{\varOmega}^4_{02} = \varOmega^4_{02}/(J_1^{-2} J_2^5),$$
$$\bar{\varOmega}^5_{10} = \varOmega^5_{10}/(J_1^{-2} J_2^4),\ \bar{\varOmega}^5_{01} = \varOmega^5_{01}/(J_1^{-3} J_2^5),\ \bar{\varOmega}^6 = \varOmega^6/(J_1^{-4} J_2^5).$$

Higher order differential invariants can be obtained in a similar way from the basic relative invariants, but alternatively we can adjust invariant derivations by letting $\nabla_j = J_1^{\rho_j} J_2^{\sigma_j} \cdot \varDelta_j|_{r=s=0}$ with a proper choice of the weights ρ_j, σ_j. Namely we let

$$\nabla_p = \frac{J_1}{J_2}\mathcal{D}_p, \qquad \nabla_x = \frac{1}{J_1}\bigl(\hat{\mathcal{D}}_x + u\mathcal{D}_p\bigr),$$
$$\nabla_y = \frac{1}{J_2}\Bigl(\mathcal{D}_y + \frac{u^5}{5u^4}\hat{\mathcal{D}}_x + \bigl(\frac{u^4_{10}}{u^4} + \frac{6u\,u^5}{5u^4} + 2u^1\bigr)\mathcal{D}_p\Bigr).$$

These form a basis of invariant derivatives over $\mathcal{I}$ and we have:

$$[\nabla_p, \nabla_x] = -\tfrac{1}{8}\bar{H}_{10}\nabla_p - \tfrac{3}{8}\bar{K}\nabla_x + \nabla_y,$$
$$[\nabla_p, \nabla_y] = (\bar{\Omega}^5_{10} - \tfrac{1}{8}\bar{H}_{01})\nabla_p + \tfrac{1}{5}\bar{\Omega}^6\nabla_x - \tfrac{1}{4}\bar{K}\nabla_y,$$
$$[\nabla_x, \nabla_y] = \bar{\Omega}^4_{20}\nabla_p + (\tfrac{1}{5}\bar{\Omega}^5_{10} + \tfrac{3}{8}\bar{H}_{01})\nabla_x - \tfrac{1}{4}\bar{H}_{10}\nabla_y.$$

The derivations and coefficients can be also expressed in terms of non-branching invariants $\tilde{J}_1 = \frac{8}{3}\nabla_x J_1$ and $\tilde{J}_2 = 4\nabla_y J_2$.

Theorem 2. *The space $\mathcal{I}$ of differential invariants is generated by the invariant derivations $\nabla_x, \nabla_y, \nabla_p$ on the generic stratum.*

Indeed, we mean here that taking coefficients of the commutators, adding their derivatives etc leads to a complete list of basic differential invariants.

On the other hand, if we want to list generators according to the order, so that invariant derivations only add new in the corresponding order, then we shall restrict to $\bar{H}_{10}, \bar{H}_{01}, \bar{K}$ in order 5, add $\bar{\Omega}^{6-i-j}_{ij}$ in order 6 and the rest in every order is generated from these by invariant derivations with ∇_j. The relations can be deduced from these of §1.2.2.

3.3 Equivalence problem

2nd order ODEs $\mathcal{E}$ can be considered as sections $\mathfrak{s}_\mathcal{E}$ of the bundle π, whence we can restrict any differential invariant $J \in \mathcal{I}_k$ to the equation via pull-back of the prolongation:

$$J^\mathcal{E} := (\mathfrak{s}^{(k)}_\mathcal{E})^*(J) \in C^\infty_{\mathrm{loc}}(\mathbb{R}^3(x, y, p)).$$

Consider most non-degenerate 2nd order ODEs $\mathcal{E}$, such that[10] $\bar{H}^\mathcal{E}_{10}$, $\bar{H}^\mathcal{E}_{01}$, $\bar{K}^\mathcal{E}$ are local coordinates on $\mathbb{R}^3(x, y, p)$. Then the other differential invariants on the equation can be expressed as functions of these:

$$\bar{H}^\mathcal{E}_{ij} = \Phi^\mathcal{E}_{ij}(\bar{H}^\mathcal{E}_{10}, \bar{H}^\mathcal{E}_{01}, \bar{K}^\mathcal{E}),\ \bar{K}^\mathcal{E}_{ij} = \Psi^\mathcal{E}_{ij}(\bar{H}^\mathcal{E}_{10}, \bar{H}^\mathcal{E}_{01}, \bar{K}^\mathcal{E}),\ \bar{\Omega}^{k\,\mathcal{E}}_{ij} = \Upsilon^{k\,\mathcal{E}}_{ij}(\bar{H}^\mathcal{E}_{10}, \bar{H}^\mathcal{E}_{01}, \bar{K}^\mathcal{E}).$$

Due to the relations above we can restrict to the following collection of functions:

$$\Phi^\mathcal{E}_{20}, \Phi^\mathcal{E}_{11}, \Phi^\mathcal{E}_{02}, \Psi^\mathcal{E}_{10}, \Psi^\mathcal{E}_{01}, \Upsilon^{6\,\mathcal{E}}, \Upsilon^{5\,\mathcal{E}}_{10}, \Upsilon^{4\,\mathcal{E}}_{20}, \tag{1}$$

the others being expressed through the given ones via the operators of derivations (which naturally restrict to $\mathcal{E}$ as directional derivatives).

Theorem 3. *Two generic 2nd order differential equations $\mathcal{E}_1$, $\mathcal{E}_2$ are point equivalent iff the collections (1.1) of functions on $\mathbb{R}^3$ coincide.*

[10] Here and in what follows one can assume (higher micro-)local treatment.

Proof. Necessity of the claim is obvious. Sufficiency is based on investigation of solvability of the corresponding Lie equation[11]

$$\mathfrak{Lie}(\mathcal{E}_1,\mathcal{E}_2)=\{[\varphi]_z^2\in J^2(\mathbb{R}^2,\mathbb{R}^2):\,\varphi^{(2)}(\mathcal{E}_1\cap\pi_{2,0}^{-1}(z))=\mathcal{E}_2\cap\pi_{2,0}^{-1}(\varphi(z))\},\quad(2)$$

which has finite type. Notice that the prolongation $\mathfrak{Lie}(\mathcal{E}_1,\mathcal{E}_2)^{(k)}$ consists of the jets $[\varphi]_z^{k+2}$ such that $\varphi^{(2)}$ transforms k-jets of the equation $\mathcal{E}_1$ to the k-jets of the equation $\mathcal{E}_2$ along the whole fiber over $z\in J^0\mathbb{R}=\mathbb{R}^2(x,y)$.

Proposition 4 *Suppose that the system $\mathfrak{Lie}(\mathcal{E}_1,\mathcal{E}_2)$ is formally solvable; more precisely let $\mathcal{T}\subset\mathfrak{Lie}(\mathcal{E}_1,\mathcal{E}_2)^{(10)}\subset J^{12}(\mathbb{R}^2,\mathbb{R}^2)$ be such a manifold that $\pi_{12}|_{\mathcal{T}}$ is a submersion onto $\mathbb{R}^2$. Then this system is locally solvable[12], so that the equations are point equivalent, i.e. $\exists\varphi\in\mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$: $\forall z\in\mathbb{R}^2$ $[\varphi]_z^2\in\mathfrak{Lie}(\mathcal{E}_1,\mathcal{E}_2)$.*

Indeed, the symbol of the system $\mathfrak{Lie}(\mathcal{E}_1,\mathcal{E}_2)$ (provided it is non-empty, which is usually the case for generic $\mathcal{E}_1,\mathcal{E}_2$) is the same as for the symmetry algebra $\mathrm{sym}(\mathcal{E})$, namely: $g_0=T=\mathbb{R}^2$, $g_1=T^*\otimes T$, $g_2\subset S^2T^*\otimes T$ has codimension 4 and no (complex) characteristic covectors, so that $g_3=g_2^{(1)}=0$, whence $\oplus\,g_i\simeq\mathrm{sym}(y''=0)\simeq\mathrm{sl}_3$.

It should be also noted that the first prolongation $\mathfrak{Lie}(\mathcal{E}_1,\mathcal{E}_2)^{(1)}\subset J^3(\mathbb{R}^2,\mathbb{R}^2)$ always exists and is of Frobenius type, while the next one has proper projection unless the compatibility conditions vanish.

We are interested in solvability of the system, so we successively add the compatibility conditions. The first belongs to the space $H^{2,2}(\mathfrak{Lie})\simeq\mathbb{R}^2$, but it may happen that only one of the components is non-zero (if both are zero, the system is compatible and we are done, if both are non-zero we have more equations to add and the process stops earlier). So we add this equation of the second order to the system of 4 equations and get a new system $\widetilde{\mathfrak{Lie}}$ of formal codim $=5$.

Then we continue to add equations-compatibilities and can do it maximum $\sum\dim g_i=8$ times, so that we get $3+8=11$-th order condition. After this we get only discrete set of possibilities for solutions and checking them we get that either we have a 12-jet solution or there do not exist solutions at all.

In these arguments we adapted dimensional count, i.e. we assumed regularity. But singularities can bring only zero measure of values (by Sard's lemma), so that our condition still works even in smooth (not only analytic) situation.

[11] It is important not to mix solvability, i.e. existence of local solutions, with compatibility, i.e. existence of solutions with all admissible Cauchy data. The latter may be cut by the compatibility conditions. This confusion occurred in the proof of Theorem 8.3 from [Y]: the Lie equation is not formally integrable except for maximally symmetric case.

[12] A regularity assumption is needed for this, which is given by the non-degeneracy condition $d\bar{H}_{10}^{\mathcal{E}}\wedge d\bar{H}_{01}^{\mathcal{E}}\wedge d\bar{K}^{\mathcal{E}}\neq 0$.

Now let us explain formal solvability for our problem. A jet $[\varphi]_z^{k+2}$ belongs to the prolongation $\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)^{(k)}$ iff $\varphi^{(k+2)}$ transforms $\mathcal{E}_1^{(k)} \cap \pi_{k+2,0}^{-1}(z)$ to $\mathcal{E}_2 \cap \pi_{k+2,0}^{-1}(\varphi(z))$. For randomly chosen equations the system $\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)$ will be empty over any point $z \in \mathbb{R}^2$ just because none map can transform the whole fiber $\mathcal{E}_1 \cap \pi_{2,0}^{-1}(z_1)$ into another fiber $\mathcal{E}_2 \cap \pi_{2,0}^{-1}(z_2)$ (example: ODEs $y'' = f(x, y, y')$ with polynomial dependence on $p = y'$ of degrees 3 and 4).

The compatibility for the system $\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)$ of order k are the conditions that φ^* transforms the restricted order k differential invariants $J^{\mathcal{E}_2}$ into $J^{\mathcal{E}_1}$. Since this is possible by our assumption, we get prolongation $\mathcal{T} \subset \mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)^{(10)}$. Moreover this $\mathcal{T}$ will be a submanifold and no singularity issues arise. This yields us local point equivalence.

Remark 1 *If differential invariants $J_1 \dots J_3$ are independent on equation $\mathcal{E}$, then there is another way to define invariant derivatives [Lie$_1$, Ol, KL$_1$], so called Tresse derivatives, which in local coordinates have the form: $\hat{\partial}/\hat{\partial} J_i = \sum_j [\mathcal{D}_a(J_b)]_{ij}^{-1} \mathcal{D}_j$. In our case, when we take $\bar{H}_{10}^{\mathcal{E}}, \bar{H}_{01}^{\mathcal{E}}, \bar{K}^{\mathcal{E}}$ as coordinates on the equation, they are just $\partial/\partial \bar{H}_{10}^{\mathcal{E}}$, $\partial/\partial \bar{H}_{01}^{\mathcal{E}}$, $\partial/\partial \bar{K}^{\mathcal{E}}$, when restricted to $\mathcal{E}$.*

Another generic case is when we have 3 functional independent invariants among[13]

$$\bar{H}_{10}^{\mathcal{E}},\ \bar{H}_{01}^{\mathcal{E}},\ \bar{K}^{\mathcal{E}},\ \bar{H}_{20}^{\mathcal{E}},\ \bar{H}_{11}^{\mathcal{E}},\ \bar{H}_{02}^{\mathcal{E}},\ \bar{K}_{10}^{\mathcal{E}},\ \bar{K}_{01}^{\mathcal{E}},\ \bar{\Omega}^{6\,\mathcal{E}},\ \bar{\Omega}_{10}^{5\,\mathcal{E}}, \bar{\Omega}_{20}^{4\,\mathcal{E}}. \tag{3}$$

In this case we can express the rest of invariants through the given 3 basic, and the classification is precisely the same as in Theorem 3.

There are other regular classes of 2nd order ODEs (in general, equations are stratified according to functional ranks):

1. Collection (1.3) has precisely 2 functionally independent invariants,
2. Collection (1.3) has only 1 functionally independent invariant,
3. Collection (1.3) consists of constants.

In cases 1 or 2 we can choose basic invariants (2 or 1 respectively – note that the space of all differential invariants, not only of collection (1.3), will then have functional rank 2 or 1) and express the rest through them. The functions-relations will be again the only obstructions to point equivalence.

In the latter case all differential invariants are constant on the equation $\mathcal{E}$, so for the equivalence these (finite number of) constants should coincide.

Remark 2 *Cartan's equivalence method provides a canonical frame (on some bundle over the original manifold), which yields all differential invariants but with mixture of orders. Otherwise around, given the algebra of differential invariants, we can choose $J_1, \dots, J_s$ among them, which are functionally independent on a generic (prolonged) equation. Then $dJ_1, \dots, dJ_s$ will be a canon-*

[13] We do not know if this is realizable in other cases, than that described by Theorem 3.

ical basis of 1-forms, which can work as a (holonomic) moving frame. Non-holonomic frames can appear upon dualizing invariant (non-Tresse) derivatives.

Let us finally give another formulation of the equivalence theorem. We can consider collection (1.3) as a map $\mathbb{R}^3 \simeq \mathcal{E} \to \mathbb{R}^{11}$ by varying the point of our equation $\mathcal{E}$. Thus we get (in regular case) a submanifold of $\mathbb{R}^{11}$ of dimension 3, 2, 1 or 0 respectively. This submanifold is an invariant (and the previous formulation was only a way to describe it as a graph of a vector-function):

Theorem 5. *Two 2nd order regular differential equations $\mathcal{E}_1$, $\mathcal{E}_2$ are point equivalent iff the corresponding submanifolds in the space of differential invariants $\mathbb{R}^{11}$ coincide.*

4 Singular stratum: projective connections

On the space $J^3\mathbb{R}^3(x, y, p)$ the lifted action of the pseudogroup $\mathfrak{h}$ is transitive. But its lift to the space of 4-jets is not longer such: There are singular strata, given by the equations $I = 0, H = 0$. Moreover they have a singular substratum $I = H = 0$ in itself, on which the pseudogroup action is transitive, so that any equation from it is point equivalent to trivial ODE $y'' = 0$ [Lie$_2$, Lio, Tr$_1$].

In this subsection we consider the singular stratum $I = 0$[14]. It corresponds to equations of the form

$$y'' = \alpha_0(x, y) + \alpha_1(x, y)p + \alpha_2(x, y)p^2 + \alpha_3(x, y)p^3, \quad p = y'. \tag{4}$$

This class of equations is invariant under point transformations. Moreover it has very important geometric interpretation, namely such ODEs correspond to projective connections on 2-dimensional manifolds [C]. We will indicate 3 different approaches to the equivalence problem.

4.1 The original approach of Tresse

The idea is to investigate the algebra of differential invariants, following S.Lie's method, and then to solve the equivalence problem via them. In [Tr$_1$] lifting the action of point transformation to the space $J^k(2, 4)$ (jets of maps $(x, y) \mapsto$

[14] The other stratum $H = 0$ can be treated similarly. Indeed, though the invariants I, H look quite unlike, they are proportional to self-dual and anti-self-dual components of the Fefferman metric [F] and this duality is very helpful [NS].

Note however that even though it is difficult to solve the PDE $H = 0$ without non-local transformations, some partial solutions can be found using symmetry methods. For instance, a 3-dimensional family of solutions is $y'' = \varphi(p)/x$ with $\varphi''' = \dfrac{\varphi''(2\varphi - 2 - \varphi')}{\varphi(\varphi - 1)}$.

$(\alpha_0,\dots,\alpha_3)$) he counts the number of basic differential invariants of pure order k to be

$$\begin{array}{l} k: 0\ 1\ 2\ 3\ 4\ 5\ \ 6\ \ \ 7\ \ \ 8\ \dots\qquad k\qquad \dots \\ \#: 0\ 0\ 0\ 0\ 6\ 8\ 10\ 12\ 14\ \dots\ 2(k-1) \end{array}$$

An independent check of this (with the same method as in §1.3.1) is given in [Y].

The action of $\mathfrak{g}$ is transitive on the space of 1st jets and its lift is transitive on the space of second jets $J^2(2,4)$ outside the singular orbit $L_1 = L_2 = 0$, where

$$L_1 = -\alpha_{2xx}+2\alpha_{1xy}-3\alpha_{0yy}-3\alpha_3\alpha_{0x}+\alpha_1\alpha_{2x}-6\alpha_0\alpha_{3x}+3\alpha_2\alpha_{0y}-2\alpha_1\alpha_{1y}+3\alpha_0\alpha_{2y}$$

$$L_2 = -3\alpha_{3xx}+2\alpha_{2xy}-\alpha_{1yy}-3\alpha_3\alpha_{1x}+2\alpha_2\alpha_{2x}-3\alpha_1\alpha_{3x}+6\alpha_3\alpha_{0y}-\alpha_2\alpha_{1y}+3\alpha_0\alpha_{3y}$$

These second order operators[15] were found by S.Lie [Lie$_2$] who showed that vanishing $L_1 = L_2 = 0$ characterizes trivial (equivalently: linearizable) ODEs. R.Liouville [Lio] proved that the tensor

$$L = (L_1dx + L_2dy) \otimes (dx \wedge dy), \tag{5}$$

responsible for this, is an absolute differential invariant.

Further on Tresse claims that all absolute differential invariants can be expressed via L_1, L_2, but [Lio, Tr$_2$] do not contain these formulae. The problem was handled recently by V.Yumaguzhin [Y] (the whole set of invariants was presented, though not fully described).

Namely it was shown that the action of $\mathfrak{g}$ in $J^3(2,4)$ is transitive outside the stratum $F_3 = 0$, where

$$\begin{aligned} F_3 = (L_1)^2\mathcal{D}_y(L_2) - L_1L_2(\mathcal{D}_x(L_2) + \mathcal{D}_y(L_1)) + (L_2)^2\mathcal{D}_x(L_1) \\ - (L_1)^3\alpha_3 + (L_1)^2L_2\alpha_2 - L_1(L_2)^2\alpha_1 + (L_2)^2\alpha_0 \end{aligned}$$

is the relative differential invariant from [Lio]. The other tensor invariants can be expressed through these. The invariant derivations are[16]

$$\nabla_1 = \frac{L_2}{(F_3)^{2/5}}\mathcal{D}_x - \frac{L_1}{(F_3)^{2/5}}\mathcal{D}_y, \quad \nabla_2 = \frac{\Psi_2}{(F_3)^{4/5}}\mathcal{D}_x - \frac{\Psi_1}{(F_3)^{2/5}}\mathcal{D}_y,$$

where

[15] corresponding to $(3k, -3h)$ in [Tr$_1$].

[16] The first one in the relative form was known already to Liouville [Lio]:

$$\tilde{\nabla}_1 = L_1\mathcal{D}_y - L_2\mathcal{D}_x + m(\mathcal{D}_x(L_2) - \mathcal{D}_y(L_1)) : \mathcal{R}^m \to \mathcal{R}^{m+2},$$

where $\mathcal{R}^m$ is the space of weight m relative differential invariants corresponding to the cocycle $C_\xi = \operatorname{div}_{\omega_0}(\xi)$, where $\omega_0 = dx \wedge dy$. He was very close, but did not write the second one.

$$\Psi_1 = -L_1(L_1)_y + 4L_1(L_2)_x - 3L_2(L_1)_x - (L_1)^2\alpha_2 + 2L_1L_2\alpha_1 - 3(L_2)^2\alpha_0,$$
$$\Psi_2 = 3L_1(L_2)_y - 4L_2(L_1)_y + L_2(L_2)x - 3(L_1)^2\alpha_3 + 2L_1L_2\alpha_2 - (L_2)^2\alpha_1.$$

Now we can get two differential invariants of order 4 as the coefficients of the commutator

$$[\nabla_1, \nabla_2] = I_1\nabla_1 + I_2\nabla_2.$$

Related invariants are the following: one applies the invariant derivations ∇_i (extended to the relative invariants) to F_3 and gets another relative differential invariant of the same weight (the relation here is almost obvious since $\nabla_1 \wedge \nabla_2$ is proportional to F_3). Thus $\nabla_1(F_3)/F_3, \nabla_2(F_3)/F_3$ are absolute invariants.

To get four more invariants $I_3, \dots, I_6$ of order 4, consider the Lie equation, formed similar to (1.2) for the cubic 2nd order ODEs (1.4), see (1.6). After a number of prolongation-projection we get a Frobenius system, and its integrability conditions yield the required differential invariants (in [Y] these are obtained in a different but seemingly equivalent way).

Now we can state that the algebra $\mathcal{I}$ is generated by the invariants $I_1, \dots, I_6$ together with the invariant derivatives ∇_1, ∇_2. An interesting problem is to describe all differential syzygies between these generators.

4.2 The second Tresse approach

The invariants of §1.2.2 are not defined on the stratum $I = 0$ due to the fact that most expressions contain I in denominator. But due to footnote[14] the relative invariants I, H are on equal footing. And in fact Tresse in [Tr_2] constructs another basis of relative invariants with H in denominator.

Thus if we restrict this set to the stratum $I = 0$ minus the trivial equation, corresponding to $I = H = 0$, we get relative/absolute differential invariants of the ODEs (1.4). For instance H is proportional to $L_1 + L_2p$, which under substitution of $p = \frac{dy}{dx}$ is proportional to the tensor L. The other invariants are rational functions in p on the cubics (1.4), which may be taken in correspondence with the invariants of the approach from §1.4.1.

The proposed idea can be viewed as a change of coordinates in the algebra $\mathcal{I}$. Yet, another approach was sketched in [Tr_1], which can be called a non-local substitution.

Namely by a point transformation Tresse achieves $L_2 = 0$, and so brings the tensor $L_1dx + L_2dy$ to the form $\lambda\, dx$. Then the point transformation pseudogroup is reduced to the triangular pseudogroup $x \mapsto X(x)$, $y \mapsto Y(x, y)$, and the invariants are generated by the invariant derivatives Δ_x, Δ_y and the invariants B, C, D of orders 1, 2, 2 respectively ([Tr_1], ch.III), which though do not correspond to the orders in the approach of §1.4.1.

4.3 Lie equations

Let $\mathfrak{s}_\mathcal{E} : \mathbb{R}^2 \to \mathbb{R}^4$ be the map $(x, y) \mapsto (a_0, a_1, a_2, a_3)$ corresponding to a 2nd order ODE $\mathcal{E}$ (1.4). With two such ODEs we relate the Lie equation on the equivalence between them:

$$\mathfrak{Lie}(\mathcal{E}_1,\mathcal{E}_2)=\{[\varphi]^2_z\in J^2(2,2):\hat{\varphi}\bigl(\mathfrak{s}_{\mathcal{E}_1}(z)\bigr)=\mathfrak{s}_{\mathcal{E}_2}(\varphi(z))\},\qquad (6)$$

where $\hat{\varphi}:\mathbb{R}^2\times\mathbb{R}^4\to\mathbb{R}^2\times\mathbb{R}^4$ is the lift of a map $\varphi:\mathbb{R}^2\to\mathbb{R}^2$ to a map of ODEs (1.4). On infinitesimal level, the lift of a vector field $X=a\,\partial_x+b\,\partial_y$ is

$$\begin{aligned}\hat{X}&=a\,\partial_x+b\,\partial_y+(b_{xx}+\alpha_0(b_y-2a_x)-\alpha_1 b_x)\,\partial_{\alpha_0}\\ &+(2b_{xy}-a_{xx}-3\alpha_0 a_y-\alpha_1 a_x-2\alpha_2 b_x)\partial_{\alpha_1}+(b_{yy}-2a_{xy}-2\alpha_1 a_y-\alpha_2 b_y-3\alpha_3 b_x)\partial_{\alpha_2}\\ &+(-a_{yy}-\alpha_2 a_y+\alpha_3(a_x-2b_y))\partial_{\alpha_3}.\end{aligned}$$

For one equation $\mathcal{E}_1=\mathcal{E}_2$ infinitesimal version of the finite Lie equation $\mathfrak{Lie}(\mathcal{E},\mathcal{E})$ describes the symmetry algebra (which more properly should be called a Lie equation [KSp]) $\mathrm{sym}(\mathcal{E})$: it is formed by the solutions of

$$\mathfrak{lie}(\mathcal{E})=\{[X]^2_z\in J^2(2,2):\hat{X}\in T_{\mathfrak{s}_{\mathcal{E}}(z)}[\mathfrak{s}_{\mathcal{E}}(\mathbb{R}^2)]\}.\qquad (7)$$

The basic differential invariants of the pseudogroup $\mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}^2)$ action on ODEs (1.4) arise as the obstruction to formal integrability of the equation $\mathfrak{lie}(\mathcal{E})$ (for the equivalence problem $\mathfrak{Lie}(\mathcal{E}_1,\mathcal{E}_2)$, but the investigation is similar). In coordinates, when the section $\mathfrak{s}_{\mathcal{E}}$ is given by four equations $\alpha_i-\alpha_i(x,y)=0$, overdetermined system (1.6) is written as

$$\begin{aligned}b_{xx}+\alpha_0(b_y-2a_x)-\alpha_1 b_x&=a\,\alpha_{0x}+b\,\alpha_{0y}\\ 2b_{xy}-a_{xx}-3\alpha_0 a_y-\alpha_1 a_x-2\alpha_2 b_x&=a\,\alpha_{1x}+b\,\alpha_{1y}\\ b_{yy}-2a_{xy}-2\alpha_1 a_y-\alpha_2 b_y-3\alpha_3 b_x&=a\,\alpha_{2x}+b\,\alpha_{2y}\\ -a_{yy}-\alpha_2 a_y+\alpha_3(a_x-2b_y)&=a\,\alpha_{3x}+b\,\alpha_{3y}\end{aligned}$$

The symbols $g_i\subset S^iT^*\otimes T$ are: $g_0=T=\mathbb{R}^2$, $g_1=T^*\otimes T\simeq\mathbb{R}^4$, $g_2\simeq\mathbb{R}^2$ and $g_{3+i}=0$ for $i\geq 0$. The compatibility conditions belong to the Spencer cohomology group $H^{2,2}(\mathfrak{lie})\simeq\mathbb{R}^2$: this is equivalent to the tensor L of (1.5). If $L=0$, the equation is integrable[17] and the solution space is the Lie algebra sl_3.

If $L\neq 0$, the equation $\mathfrak{lie}_0=\mathfrak{lie}(\mathcal{E})$ has prolongation-projection[18] $\mathfrak{lie}_1=\pi_{4,1}(\mathfrak{lie}^{(2)})$ with symbols $g_0=T$, $\bar{g}_1\simeq\mathbb{R}^2\subset g_1$, $g_2\simeq\mathbb{R}^2$ and $g_{3+i}=0$ for $i\geq 0$.

After prolongation-projection, one gets the equation $\mathfrak{lie}_2$ with symbols $g_0=T$, $\tilde{g}_1\simeq\mathbb{R}^1\subset\bar{g}_1$ and $g_{2+i}=0$ for $i\geq 0$. This equation has the following space of compatibility conditions: $H^{1,2}(\mathfrak{lie}_3)\simeq\mathbb{R}^1$. It yields the condition of the third order in α_i: $F_3=0$ (this, together with other invariants [R], characterizes equations with 3-dimensional symmetry algebra, namely sl_2).

[17] not only formally, but also locally smoothly due to the finite type of $\mathfrak{lie}$.

[18] This means that the Lie equation has the first prolongation $\mathfrak{lie}^{(1)}\subset J^3(2,2)$, but the next prolongation exists only over the jets of vector fields X, preserving the tensor L.

If $F_3 \neq 0$, then the prolongation-projection yields the equation $\mathfrak{lie}_3$ with $g_0 = T$ and $g_{1+i} = 0$ for $i \geq 0$. The compatibility conditions are given by the Frobenius theorem and this provides the basis of differential invariants.

Remarks. 1. The idea to reformulate equivalence problem via solvability of an overdetermined system appeared in S.Lie's linearization theorem, where he showed that an ODE (1.4) is point equivalent to the trivial equation $y'' = 0$ iff the system (see [Lie$_2$], p.365 (we let $z = c, w = C$ etc), and also [IM])

$$\frac{\partial w}{\partial x} = zw - \alpha_0\alpha_3 - \frac{1}{3}\frac{\partial \alpha_1}{\partial y} + \frac{2}{3}\frac{\partial \alpha_2}{\partial x}, \qquad \frac{\partial z}{\partial x} = z^2 - \alpha_0 w - \alpha_1 z + \frac{\partial \alpha_0}{\partial y} + \alpha_0\alpha_2,$$
$$\frac{\partial w}{\partial y} = -w^2 + \alpha_2 w + \alpha_3 z + \frac{\partial \alpha_3}{\partial x} - \alpha_1\alpha_3, \quad \frac{\partial z}{\partial y} = -zw + \alpha_0\alpha_3 - \frac{1}{3}\frac{\partial \alpha_2}{\partial x} + \frac{2}{3}\frac{\partial \alpha_1}{\partial y}.$$

is compatible. The compatibility conditions here are given by the Frobenius theorem: $L_1 = L_2 = 0$. In fact, the system can be transformed into a linear system[19], which is equivalent to half of our once prolonged Lie equation $\mathfrak{lie}^{(1)}$ (Lie considers combinations of the unknown functions-component of the point transformation, that's why in the third order we get only $4 = 8/2$ equations, the second half of equations was not much used by him).

2. Other ways of getting differential invariants arise from problems which have projectively invariant answers. For instance the following system arose in 3 independent problems:

$$u_y = P_0[u, v, w], \;\; u_x + 2v_y = P_1[u, v, w], \;\; 2v_x + w_y = P_2[u, v, w], \;\; w_x = P_3[u, v, w],$$

where $P_i[u, v, w]$ are linear operators of a special type, with coefficients being smooth functions in x, y. This system can be obtained similar to $\mathfrak{lie}$ from the condition of existence of Killing tensors[20].

In [K] solvability of this system lead to an invariant characterization of Liouville metrics, in [BMM] to normal forms of metrics with transitive group of projective transformations and in [BDE] – to the condition of local metrisability of projective structures on surfaces.

All these problems have the answers (for instance, in the first mentioned paper, the number of Killing tensors of a metric), which are projective invariants. Thus they provide projective differential invariants and in turn can be expressed via any basis of them.

3. Many papers addressed the higher-dimensional version of the same equivalence problem (which is surprisingly easier, because the Lie equation is more overdetermined). In Cartan [C] this is the study of the projective connection. Refs. [Th, Lev] address the algebra of scalar projective differential invariants.

[19] S.Lie considers finite transformations, whence the non-linearity. A projective transformation is needed to change this into a linear system, while the infinitesimal analog — our Lie equation $\mathfrak{lie}(\mathcal{E})$ – is linear from the beginning.

[20] Substitution $u = 3\xi_y, w = 3\eta_x, v = -(\xi_x + \eta_y)/2$ transforms this system to the kind $\xi_{yy} = \dots, 2\xi_{xy} - \eta_{yy} = \dots, \xi_{xx} - 2\eta_{xy} = \dots, -\eta_{xx} = \dots$, which has the same symbol as (1.7).

However in neither of these approaches the Tresse method was superseded. For instance, in the latter reference even the number of differential invariants for the 2-dimensional case was not determined. On the other hand, the method of Lie equations allows to obtain the algebra of projective invariants in the higher-dimensional case as well.

5 Application to symmetries

At the end of [Tr_2] a classification of symmetric equations is given. It turns out that the symmetry algebra can be of dimensions 8, 3, 2, 1 or 0. This follows from the study of dependencies among differential invariants, and it is not obvious that this automatically applies to all singular strata (but it is true).

Thus if $\dim \mathrm{Sym}(\mathcal{E}) = 8$ the ODE is equivalent to the trivial $y'' = 0$. If $\dim \mathrm{Sym}(\mathcal{E}) = 3$, the normal forms are ($y' = p$):

$$\begin{array}{ll} y'' = p^a & y'' = \dfrac{(cp + \sqrt{1-p^2})(1-p^2)}{x} \\ y'' = e^p & y'' = \pm(xp - y)^3. \end{array}$$

Only the last form belongs to the singular stratum $I = 0$.

Due to symmetry between I and H, there should be corresponding normal form with $H = 0$. Here one can be mislead since direct calculations shows that none has vanishing H. The reason is however that Tresse uses Lie's classification of Lie algebras representation by vector fields on the plane. For 3-dimensional algebras Lie used normal forms over $\mathbb{C}$, and ineed the third normal form has $H = 0$ for the parameter $c = \pm i$. Thus over $\mathbb{R}$ the above normal forms should be extended.

As the symmetry algebra reduces to dimensions 2 we have the respective normal forms

$$y'' = \psi(p) \qquad \text{and} \qquad y'' = \psi(p)/x.$$

It is important that for singular strata the classification shall be finer. This is almost obvious for projective connections (cubic ψ), but for metric projective connections this is already substantial, see [BMM].

The case $\dim \mathrm{Sym}(\mathcal{E}) = 1$ has only one quite general form $y'' = \psi(x, p)$ with an obvious counterpart for projective connections (for the metric case see [Ma]).

Remark 3 *When the transformation pseudogroup reduces from point to fiber-preserving (triangular) transformations of $J^0\mathbb{R}(x) = \mathbb{R}^2(x, y)$, the algebra of differential invariants grows, but the symmetric cases change completely. In particular, the symmetry algebra can have dimensions 6, 3, 2, 1 or 0 [KSh, HK].*

Not much is known about the criteria for having the prescribed dimension of the symmetry algebra, except for the corollary of Lie-Liouville-Tresse theorem: $\dim \mathrm{Sym}(\mathcal{E}) = 8$ iff $L = 0$.

In [Tr_2] the following was claimed (we translate it to the language of absolute differential invariants):

- ◇ The symmetry algebra is 3-dimensional iff all the differential invariants (on the equation) are constant.
- ◇ The symmetry algebra is 2-dimensional iff the space of differential invariants has functional rank 1, i.e. any two of them are functionally dependent (Jacobian vanishes).
- ◇ The symmetry algebra is 1-dimensional iff the space of differential invariants has functional rank 2 (all 3×3 Jacobians vanish).

This (unproved in Tresse, but correct statement) is however inefficient, since checking all invariants is not practically possible. Here is an improvement:

Theorem 6. *The above claims hold true if we restrict to the basic absolute differential invariants of order* ≤ 6 *described in §1.2.2.*

What is the minimal collection of differential invariants answering the above question is seemingly unknown (except for the case $\dim \mathrm{Sym}(\mathcal{E}) = 3$ for $I = 0$ handled in [R]).

A Appendix: Another approach

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1. Consider the stabilizer $\mathfrak{l}_8 \subset \mathfrak{g}$ of a point $(x, y) \in \mathbb{R}^2(x, y) = J^0\mathbb{R}(x)$. We can choose coordinates so that $x = y = 0$. The vector fields generating this subalgebra of vector fields of $F_{x,y} = \mathbb{R}^2(p, u) = \pi_{2,0}^{-1}(x, y)$ are

$$\mathfrak{l}_8 = \langle \partial_p,\ \partial_u,\ p\,\partial_p,\ u\,\partial_u,\ p\,\partial_u,\ p^2\partial_u,\ p^3\partial_u,\ p^2\partial_p + 3p\,u\,\partial_u \rangle$$

This is an 8-dimensional Lie algebra with Levi decomposition $\mathcal{R}_5 \ltimes \mathrm{sl}_2$, where $\mathcal{R}_5$ is the radical, which is a solvable Lie algebra with 4-dimensional (commutative) nil-radical.

In $F_{x,y}$ the 2nd order equation $\mathcal{E}$ is a curve[21] $u = f(p)$. Since in equivalence problem we can transform one base point to another by a point transformation (any point to any if the equation possesses a 2-dimensional symmetry group, transitive on the base), the equivalence problem is reduced to the equivalence of curves on the plane $\mathbb{R}^2(p, u)$ with respect to the Lie group $\mathfrak{l}_8$ action.

The action lifts to the spaces $J^k\mathbb{R}(p) = \mathbb{R}^{k+2}(p, u, u_p, \dots)$ and is transitive up to 3rd jets. The first singular orbit appears in the space $J^4\mathbb{R}(p)$ and is $\mathcal{S}_1 = \{G_4 = u^4 = 0\}$ (we continue to use the same notations as above, so that

[21] depending parametrically on x, y.

$u^4 = u_{ppppp}$). The next singular orbit (different from prolongation of this one) appears in the space $J^6\mathbb{R}(p)$ and is $\mathcal{S}_2 = \{G_6 = 5u^4u^6 - 6u^5 \cdot u^5 = 0\}$.

Notice that the second equation belongs to the prolongation of the first: $\mathcal{S}_2 \subset \mathcal{S}_1^{(2)}$ (so it is a sub-singular orbit). The functions G_4, G_6 are relative differential invariants. In this case the weights can be chosen via cocycles $C_\xi^r = \operatorname{div}_{\omega_0}(\xi) - \frac{1}{2}\operatorname{div}_\Omega(\xi_1)$ and $C_\xi^s = \frac{1}{2}\operatorname{div}_\Omega(\xi_1)$, where $\omega_0 = dp \wedge du$, $\Omega = -\omega \wedge d\omega = dp \wedge du \wedge du_p$ (for $\omega = du - u_p\, dp$) and $\xi = A\partial_p + B\partial_u$, $\xi_1 = X_f = A\partial_p + B\partial_u + (\mathcal{D}_p(B) - \partial_p(A)u_p)\partial_{u_p}$, $\hat{\xi} = X_f^{(\infty)}$ with

$$f = B - A\,u_p, \quad A = a_0 + a_1p + a_2p^2, \quad B = b_0 + b_1p + b_2p^2 + b_3p^3 + b_4u + 3a_2p\,u.$$

Denoting $\mathcal{R}^{r,s} = \{\psi \in C^\infty(J^\infty\mathbb{R}) : \hat{\xi}(\psi) = -(r\,C_\xi^r + s\,C_\xi^s)\psi\}$, we get[22]:

$$G_4 \in \mathcal{R}^{4,-1},\ G_6 \in \mathcal{R}^{10,-2}.$$

The relative invariant derivative here equals

$$\diamondsuit_p = \mathcal{D}_p - \frac{2r+3s}{5}\frac{u^5}{u^4} : \mathcal{R}^{r,s} \to \mathcal{R}^{r+1,s}.$$

It acts trivially on G_4, but from its action on G_6 we can extract an absolute invariant. Indeed since $G_4/\sqrt{G_6} \in \mathcal{R}^{-1,0}$, we have: $\diamondsuit_p(G_4/\sqrt{G_6}) \in \mathcal{R}^{0,0} = \mathcal{I}$ and the latter expression is non-zero.

Actually the action of our 8-dimensional group has open orbits through generic points on $J^6\mathbb{R}(p)$ and the first absolute differential invariant appear at order 7 and equals[23]

$$I_7 = \frac{25(u^4)^2u^7 + 84(u^5)^3 - 105u^4u^5u^6}{(G_6)^{3/2}},$$

which coincides with $-10\,\diamondsuit_p(G_4/\sqrt{G_6})$.

In each higher order we get 1 new differential invariant. They determine Tresse derivative (see [KL$_1$]), but we can obtain the absolute invariant derivative directly:

$$\square_p = \frac{G_4}{\sqrt{G_6}}\diamondsuit_p\Big|_{r=s=0} = \frac{u^4}{\sqrt{5u^4u^6 - 6u^5 \cdot u^5}}\,\mathcal{D}_p : \mathcal{I} \to \mathcal{I}.$$

This can be expressed via invariants of §1.3.2 as $\frac{1}{\sqrt{5\Omega^6}}\nabla_p$.

Thus on the generic stratum every differential invariant is (micro-locally) a function of the invariants $I_7, \square_p(I_7), \square_p^2(I_7), \dots$ of orders $7, 8, 9, \dots$

2. It is easy to see that the class of cubic curves $u = Q_3(p)$ is invariant with respect to the Lie group $\mathfrak{L}_8 = \operatorname{Exp}(\mathfrak{l}_8)$ action. This 4-dimensional space forms a singular orbit, on which $\mathfrak{L}_8$ acts transitively.

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[22] Note that since the group is changed the grading is changed as well. In particular G_4, which formally coincides with I of section §1.2.1, has a different grading.

[23] Superscript after brackets means the power, while the others are indices.

Another singular orbit is $\mathcal{S}_2$, which is a 6-dimensional manifold of curves $u = (a_0 + a_1 p + a_2 p^2 + a_3 p^3) + b/(p - c)$, and $\mathfrak{L}_8$ acts transitively there on the generic stratum. The singular stratum is given by the equation $b = 0$ and coincides with the previous stratum $\mathcal{S}_1$.

3. Consider the stabilizer $\mathfrak{l}_6 = \mathrm{St}_0 \subset \mathfrak{h} = \mathfrak{g}_2$ of a point $x_2 = (x, y, p, u) \in J^0\mathbb{R}^3(x, y, p) = J^2\mathbb{R}(x)$. Since the pseudogroup $\mathfrak{g}_2$ acts transitively on $J^2\mathbb{R}(x)$, choice of x_2 is not essential, in particular we can take a coordinate representative $(x, y, 0, 0)$. This Lie algebra is generated by (prolongation of) the fields

$$\mathfrak{l}_6 = \langle p\,\partial_p,\ u\,\partial_u,\ p\,\partial_u,\ p^2\partial_u,\ p^3\partial_u,\ p^2\partial_p + 3p\,u\,\partial_u\rangle$$

and is solvable (with 4-dimensional nil-radical of length 2). Thus investigation of its invariants is easier thanks to S.Lie quadrature theorem.

Moreover we can continue and take the stabilizer $\mathrm{St}_3 \subset \mathfrak{g}_5$ of a point in $J^3\mathbb{R}^3(x, y, p)$, where the action of $\mathfrak{h}$ is still transitive. This stabilizer is a trivial 1-dimensional extension of the 2D solvable Lie algebra.

4. With all these approaches we get enough invariants to pursue classification in generic case (and even to deal with singular orbits). Indeed, we get a subalgebra $\mathfrak{V}$ in the algebra of all differential invariants $\mathcal{I}$, which we can restrict to the equation. Provided that there are invariants in $\mathfrak{V}$ independent, on our 2nd order ODE $\mathcal{E}$, we solve the equivalence problem.

In general to restore the whole algebra $\mathcal{I}$ we must add to this vertical invariants algebra $\mathfrak{V}$ invariant derivatives ∇_x, ∇_y. This is similar to Liouville approach for cubic 2nd order ODEs (retrospectively after [C] – projective connections), who found in [Lio] only a subalgebra of (relative) differential invariants (the second relevant invariant derivative $\nabla_2 = \Psi_2\mathcal{D}_x - \Psi_1\mathcal{D}_y + \dots : \mathcal{R}^r \to \mathcal{R}^{r+4}$ and the corresponding part of differential invariants was not established).

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