On classi…cation of a system of non-linear …rst order PDE's of Jacobi type

by

Ole Kristian Fossum

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Signature of Author . . . .
Department of Mathematics
1 December 2002

Certi…ed by . . . .
.
.
Research Head

Certi…ed by . . . .
Valentin Lychagin
Professor
Thesis Supervisor

Accepted by . . . .
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.

Contents

Notation

- $\mathbb{R}$ real numbers
- V vector space
- $\mathbb{C} \otimes_{\mathbb{R}} V$ is the complexification of the $\mathbb{R}$-vector space V
- $\Lambda^i(V)$ the skew-symmetric i-form on a vector space V
- $S^i(V)$ the symmetric i-form on a vector space V
- M, N manifolds
- $f : M \longrightarrow N$ a smooth map
- $f^* : C^\infty(N) \longrightarrow C^\infty(M)$
- $\Lambda^2(f^*)\,\theta(X,Y)$ is the pullback of the 2-form θ, $\Lambda^2(f^*)\,\theta(X,Y) = \theta(fX, fY)$
- $T_x^*(M)$ cotangent space of M at the point x
- $T_x(M)$ tangent space of M at the point x
- $T^*(M) = \bigcup T_x^*(M)$ the total space of the cotangent bundle
- $T(M) = \bigcup T_x(M)$ the total space of the cotangent bundle
- $\tau_M^* : T^*(M) \longrightarrow M$ the cotangent bundle of M
- $\tau_M : T(M) \longrightarrow M$ the tangent bundle of M
- $\Lambda^i(\tau_M^*)$ the i-th exterior power of the cotangent bundle of M
- $\Lambda^i(\tau_M)$ the i-th exterior power of the tangent bundle of M
- $\Omega^i(M)$ the module of sections of $\Lambda^i(\tau_M^*)$
- $\mathcal{D}^i(M)$ the module of sections of $\Lambda^i(\tau_M)$

Chapter 1

Introduction

In this thesis we will investigate a system of two non-linear first-order PDE's of Jacobi type. The PDE system will have the form:

$$\begin{cases} a_1 + b_1\frac{\partial h_1}{\partial x_1} - c_1\frac{\partial h_1}{\partial x_2} - d_1\frac{\partial h_2}{\partial x_2} + e_1\frac{\partial h_2}{\partial x_1} + f_1\left(\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right) = 0 \\ a_2 + b_2\frac{\partial h_1}{\partial x_1} - c_2\frac{\partial h_1}{\partial x_2} - d_2\frac{\partial h_2}{\partial x_2} + e_2\frac{\partial h_2}{\partial x_1} + f_2\left(\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right) = 0 \end{cases}, \qquad (*)$$

where x_1, x_2 are the independent variables, h_1, h_2 are the unknown functions and $a_i, ..., f_i$ are functions of x_1, x_2, h_1, h_2.

The system consists of two quasi-linear first-order PDE's, together with the two non-linear terms $\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}$.

The term $\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}$ is the Jacobian-determinant of $h = (h_1, h_2)$, with respect to x_1, x_2, and thereby, the name "system of Jacobi type".

From now on we will refer to the PDE system as the **Jacobi PDE system**.

By adding the Jacobi terms to the quasi-linear system, we get a class of PDE systems which is invariant with respect to coordinate transformations. In other words, if we take a Jacobi PDE system and apply any coordinate transformations, we will still have a Jacobi PDE system.

To illustrate this, we apply a hodograph transformation to the Cauchy-Riemann system.

Let x_1, x_2 be the independent variables, and h_1, h_2 the unknown functions. Then the Cauchy-Riemann system is given by:

$$\begin{cases} \frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2} = 0 \\ \frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2} = 0 \end{cases} .$$

The hodograph transformation ϕ which we will perform, is simply changing one of the dependant variables with one of the independent variables:

$$(x_1, x_2, u_1, u_2) \overset{\phi}{\mapsto} (x_1, u_1, x_2, u_2) .$$

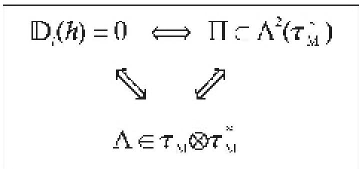
One can check that after the hodograph transformation ϕ, the Cauchy-Riemann system will have the form:

$$\begin{cases} \frac{\partial \tilde{h}_1}{\partial \tilde{x}_2}\frac{\partial \tilde{h}_2}{\partial \tilde{x}_1} - \frac{\partial \tilde{h}_1}{\partial \tilde{x}_1}\frac{\partial \tilde{h}_2}{\partial \tilde{x}_2} = 1 \\ \frac{\partial \tilde{h}_1}{\partial \tilde{x}_1} - \frac{\partial \tilde{h}_2}{\partial \tilde{x}_2} = 0 \end{cases} .$$

Clearly, it is not a quasi-linear PDE system, but a Jacobi PDE system.

As we can see, the class of quasi-linear PDE systems is not closed with respect to transformation of known and unknown variables.

The logical scheme of this thesis is given by the following diagram:



Explanation to the figure:

$\mathbb{D}_i(h) = 0$ in the diagram is the two PDE's $(i = 1, 2)$ in the Jacobi PDE system.

$\Pi \subset \Lambda^2(\tau^*_M)$ Π is a 2-dimensional subbundle in the bundle of differential 2-forms.

A is a smooth field of operators on M.

1) The top line: $\mathbb{D}_i(h) = 0 \iff \Pi \subset \Lambda^2(\tau^*_M)$, illustrates that the Jacobi PDE system can equally be represented as a 2-dimensional subbundle in the bundle of differential 2-forms. This equivalence is described in chapter (2).

2) The right line: $\Pi \subset \Lambda^2(\tau_M^*) \Longleftrightarrow A \in \tau_M \otimes \tau_M^*$, illustrates that the 2-dimensional subbundle Π in the bundle of differential 2-forms, can equally be represented as a smooth field A of operators on M. The implication from left to right is described in chapter (4), while the implication from right to left is described in chapter (5).

3) The left line $\mathbb{D}_i(h) = 0 \Longleftrightarrow A \in \tau_M \otimes \tau_M^*$, is a direct consequence of 1) and 2).

These relations will enable us to make a pointwise classification of Jacobi PDE systems. The classification is invariant with respect to coordinate transformations.

The classification is different from the standard classification of elliptic, hyperbolic and parabolic types, and depends on all of the functions $a_i, ..., f_i$.

The main result in this thesis is a necessary and sufficient criterion for when an elliptic or hyperbolic Jacobi PDE system can be transformed into the Cauchy-Riemann system or the Wave system.

Not only do we give the criterion, this thesis also provides a constructive way of finding it for a given Jacobi PDE system.

All the structures dealt with in this thesis are smooth, and if not stated otherwise, they should be regarded as smooth.

Chapter 2

Jacobi PDE system

2.1 Representation of the Jacobi PDE system by two differential 2-forms

In this section we will discuss a representation of the Jacobi PDE system. We will show how to represent the system by a 2-dimensional subbundle in the bundle of differential 2-forms over a 4-dimensional manifold. This description will enable us to make pointwise classification of the Jacobi PDE system, which is the preface of our analysis of the Jacobi PDE system. The classification will be described in the next chapter.

Let us consider the arithmetic space $\mathbb{R}^4$ with coordinates (x_1, x_2, u_1, u_2).

Let h be a smooth map from $\mathbb{R}^2$ to $\mathbb{R}^2$, such that:

$$\begin{aligned} h : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2, \\ (x_1, x_2) &\longmapsto (h_1(x_1, x_2), h_2(x_1, x_2)) \,. \end{aligned}$$

The graph of h determines a two-dimensional submanifold $\mathcal{L}_h$ in $\mathbb{R}^4$:

$$\begin{aligned} graph(h) &= \mathcal{L}_h \subset \mathbb{R}^2 \times \mathbb{R}^2, \\ \mathcal{L}_h &= \left\{ (x_1, x_2, u_1, u_1) \in \mathbb{R}^4 | \; u_1 = h_1(x_1, x_2) \text{ and } u_2 = h_2(x_1, x_2) \right\} . \end{aligned}$$

Notation 2.1 *In order to simplify the expressions, we will make the following assumptions:*

$$x = (x_1, x_2)\,,\ \ u = (u_1, u_1)\ \ \textit{and}\ h = (h_1(x_1, x_2), h_2(x_1, x_2))\,.$$

We have the following important observation $[Ly2]$: any smooth 2-form $\omega \in \Omega^2\left(\mathbb{R}^4\right)$ determines the first order non-linear differential operator $\Delta_\omega : C^\infty\left(\mathbb{R}^2, \mathbb{R}^2\right) \longrightarrow \Omega^2(\mathbb{R}^2)$, in the following way:

$$h \longmapsto \Delta_\omega(h) = \omega|_{graph(h)}.$$

To find the coordinate form of this operator, let us assume that ω is given by:

$$\begin{aligned} \omega \;=\;& a(x,u)dx_1 \wedge dx_2 \;+ b(x,u)du_1 \wedge dx_2 \;+ c(x,u)du_1 \wedge dx_1 \\ & +d(x,u)du_2 \wedge dx_1 + e(x,u)du_2 \wedge dx_2 \;+ f(x,u)du_1 \wedge du_2\,. \end{aligned}$$

Then we get:

$$\begin{aligned} \Delta_\omega \;:\;& h \mapsto a(x,h)dx_1 \wedge dx_2 + b(x,h)\frac{\partial h_1}{\partial x_1}dx_1 \wedge dx_2 + c(x,h)\frac{\partial h_1}{\partial x_2}dx_2 \wedge dx_1 \\ & +d(x,h)\frac{\partial h_2}{\partial x_2}dx_2 \wedge dx_1 + e(x,h)\frac{\partial h_2}{\partial x_1}dx_1 \wedge dx_2 \\ & +f\,(x,h)\left(\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right)dx_1 \wedge dx_2. \end{aligned}$$

We also note that Δ_ω is $C^\infty\left(\mathbb{R}^4\right)$-linear in the ω-argument.

In other words, the differential equation:

$$\begin{aligned} \mathbb{D}\,(h) \;=\;& a + b\frac{\partial h_1}{\partial x_1} - c\frac{\partial h_1}{\partial x_2} - d\frac{\partial h_2}{\partial x_2} + \\ & e\frac{\partial h_2}{\partial x_1} + f\left(\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right) = 0, \end{aligned}$$

can be represented by the 2-form ω :

$$\begin{aligned}\omega &= a(x,u)dx_1 \wedge dx_2 \ + b(x,u)du_1 \wedge dx_2 \ + c(x,u)du_1 \wedge dx_1 \\ &\quad +d(x,u)du_2 \wedge dx_1 + e(x,u)du_2 \wedge dx_2 \ + f(x,u)du_1 \wedge du_2 \ ,\end{aligned}$$

in the sense that:

$$\mathbb{D}\left(h\right) = 0 \Longleftrightarrow \omega|_{\mathcal{L}_h} = 0.$$

So, to any system $\mathbb{D}_1\left(h\right) = 0,\ \mathbb{D}_2\left(h\right) = 0$, we may use the correspondence above, and get two 2-forms ω_1 and ω_2, such that h satisfies the above equations if and only if $\omega_1|_{\mathcal{L}_h} = 0$ and $\omega_2|_{\mathcal{L}_h} = 0$. Clearly, we are allowed to take linear combinations of the equations $\left(\mathbb{D}_1\left(h\right) = 0,\ \mathbb{D}_2\left(h\right) = 0\right)$, with coefficients being smooth functions, in a non-degenerate way, without altering the system.

$$\left\{\begin{array}{l} a_{11}\mathbb{D}_1\left(h\right) + a_{12}\mathbb{D}_2\left(h\right) = 0 \\ a_{21}\mathbb{D}_1\left(h\right) + a_{22}\mathbb{D}_2\left(h\right) = 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \bar{\omega}_1 = a_{11}\omega_1 + a_{12}\omega_2 \\ \bar{\omega}_2 = a_{21}\omega_1 + a_{22}\omega_2 \end{array}\right\}, \text{ where } |a_{ij}| \in C^\infty(\mathbb{R}^4),$$

such that $\det||a_{ij}|| \neq 0$.

Hence, for a system $\mathbb{D}_1\left(h\right) = 0,\ \mathbb{D}_2\left(h\right) = 0$, one can identify a smooth field of 2-dimensional subspaces $\Pi : x \in M \longmapsto \Pi\left(x\right) \subset \Lambda^2\left(T_x^*\mathbb{R}^4\right)$, where $\Pi\left(x\right)$ is generated by $\omega_{1,x}$ and $\omega_{2,x}$. Or, in other words, with a 2-dimensional subbundle Π in the bundle $\Lambda^2\left(\tau^*_{\mathbb{R}^4}\right)$.

The construction above gives rise to the following definitions:

Definition 2.2 *Any smooth field* Π *of 2-dimensional planes:*

$$\begin{aligned}\Pi : M &\longrightarrow \Lambda^2(T^*M), \\ x &\longmapsto \Pi(x) \subset \Lambda^2(T_x^*M),\end{aligned}$$

on any 4-dimensional manifold M*, will be called a Jacobi PDE system on 2-dimensional submanifold of* M*.*

Definition 2.3 *A 2-dimensional submanifold $L \subset M$, will be called a **solution** of Π, if:*

$$\omega_x|_{T_xL} = 0 \text{ for any } x \in L \text{ , and for any } \omega_x \in \Pi(x).$$

Definition 2.4 *Let Π be a Jacobi PDE system. Then at a point x, the plane $\Pi(x) = \langle \omega_{1,x}, \omega_{2,x} \rangle$, will be called a **Jacobi plane**.*

Definition 2.5 *Let V be a 4-dimensional vector space. Then a plane Π will be called a Jacobi plane.*

The plane $\Pi(x) \subset \Lambda^2(T_x^*M)$, we call a Jacobi plane at the point x.

In other words, the Jacobi PDE system is a smooth family of Jacobi planes.

So, we managed to give a description of the Jacobi PDE system, namely to that of a 2-dimensional subbundle in $\Lambda^2(\tau_M^*)$.

This completes the upper equivalence in the triangle.

2.1.1 Symmetries and conservation laws for the Jacobi PDE systems

Let $\Pi \subset \Lambda^2(\tau_M^*)$ be a Jacobi PDE system.

Definition 2.6 *By a **symmetry** of the Jacobi PDE system, we mean a diffeomorphism $F : M \longrightarrow M$, such that $\Lambda^2(F^*) : \Pi \longrightarrow \Pi$. That is:*

$$\begin{aligned} \Lambda^2 F^*(\omega_1) &= a_{11}\omega_1 + a_{12}\omega_2, \\ \Lambda^2 F^*(\omega_2) &= a_{21}\omega_1 + a_{22}\omega_2, \end{aligned}$$

for some $a_{ij} \in C^\infty(M)$, and any basis ω_1, ω_2 on Π.

Proposition 2.7 *If F is a symmetry of Π, and L is a solution of Π, then $F(L)$ is a solution of Π.*

Proof. Since L is a solution, we know that: $\omega_1|_L = 0$ and $\omega_2|_L = 0:$

$$\Lambda^2 F^* (\omega_1)|_L = \Lambda^2 F^* (\omega_1|_L) = 0,$$

and so,

$$\Lambda^2 F^* (\omega_1)|_L = a_{11}\omega_1|_{F(L)} + a_{12}\omega_2|_{F(L)} = 0.$$

Similarly, we get:

$$\Lambda^2 F^* (\omega_2)|_L = \Lambda^2 F^* (\omega_2|_L) = 0$$

and so,

$$\Lambda^2 F^* (\omega_2)|_L = a_{21}\omega_1|_{F(L)} + a_{22}\omega_2|_{F(L)} = 0.$$

Thus, we conclude that $F(L)$ is a solution of Π, if F is a symmetry of Π. ■

Definition 2.8 *An **infinitesimal symmetry** of the Jacobi PDE system* $\Pi = \langle \omega_1, \omega_2 \rangle$, *is a smooth vector field* $X \in \mathcal{D}(M)$, *such that:*

$$\begin{aligned} L_X(\omega_1) &= a_{11}\omega_1 + a_{12}\omega_2, \\ L_X(\omega_2) &= a_{21}\omega_1 + a_{22}\omega_2, \end{aligned}$$

for some $a_{ij} \in C^\infty(M)$.

Here $L_X(\omega)$ *is the Lie derivative of* ω *along the vector field* X.

Remark 2.9 *Let* F_t *be a 1-parametric group of symmetries, then* $X = \frac{dF_t^*}{dt}|_{t=0}$ *is a infinitesimal symmetry.*

Proposition 2.10 *Let* $X \in \mathcal{D}(M)$ *be an infinitesimal symmetry, and let* F_t *be the corresponding 1-parametric group. Then* F_t *is a symmetry of* Π, *for any* t.

Proof. Denote $\Lambda^2 F_t^*(\omega_1) = \omega_{1t}$ and $\Lambda^2 F_t^*(\omega_2) = \omega_{2t}$.

Since:

$$\frac{d}{dt} F_t^*(\theta)|_{t=t_0} = F_{t_0}^*(L_X(\theta)),$$

we get:

$$\begin{cases} \frac{d\omega_{1t}}{dt} = \Lambda^2 F_t^* \left(L_x \omega_1\right) = F_t^* \left(a_{11}\right) \omega_{1t} + F_t^* \left(a_{12}\right) \omega_{2t} \\ \frac{d\omega_{2t}}{dt} = \Lambda^2 F_t^* \left(L_x \omega_2\right) = F_t^* \left(a_{21}\right) \omega_{1t} + F_t^* \left(a_{22}\right) \omega_{2t} \end{cases}.$$

Denote $F_t^* \left(a_{ij}\right) = A_{ij}\left(t\right)$, then:

$$\begin{cases} \frac{d\omega_{1t}}{dt} = A_{11}\left(t\right) \omega_{1t} + A_{12}\left(t\right) \omega_{2t} \\ \frac{d\omega_{2t}}{dt} = A_{21}\left(t\right) \omega_{1t} + A_{22}\left(t\right) \omega_{2t} \end{cases}.$$

Hence:

$$\frac{d\omega_t}{dt} = A\left(t\right) \omega_t, \tag{2.1}$$

where $||A_{ij}\left(t\right)|| = A\left(t\right)$ and $\omega_t = \begin{pmatrix} \omega_{1t} \\ \omega_{2t} \end{pmatrix}$.

Equation (2.1), is a homogenous two dimensional linear ODE system, with the initial conditions $\omega_{10} = \omega_1$ and $\omega_{20} = \omega_2$.

Let us consider an auxiliary ODE system:

$$\begin{cases} \frac{dx_1}{dt} = A_{11}\left(t\right) x_1 + A_{12}\left(t\right) x_2 \\ \frac{dx_2}{dt} = A_{21}\left(t\right) x_1 + A_{22}\left(t\right) x_2 \end{cases}.$$

Denote the fundamental matrix by T_t of the auxiliary ODE system, in the sense that:

$$\begin{bmatrix} x_1\left(t\right) \\ x_2\left(t\right) \end{bmatrix} = T_t \begin{bmatrix} x_1\left(0\right) \\ x_2\left(0\right) \end{bmatrix},$$

where:

$$T_t = \begin{bmatrix} B_{11}\left(t\right) & B_{12}\left(t\right) \\ B_{21}\left(t\right) & B_{22}\left(t\right) \end{bmatrix}.$$

Then:

$$\begin{cases} \omega_{1t} = B_{11}\left(t\right) \omega_1 + B_{12}\left(t\right) \omega_2 \\ \omega_{2t} = B_{21}\left(t\right) \omega_1 + B_{22}\left(t\right) \omega_2 \end{cases},$$

is the only solution of (2.1).

Since:

$$\begin{aligned}\Lambda^2 F_t^* (\omega_1) &= \omega_{1t} = B_{11}(t)\,\omega_1 + B_{12}(t)\,\omega_2,\\ \Lambda^2 F_t^* (\omega_2) &= \omega_{2t} = B_{21}(t)\,\omega_1 + B_{22}(t)\,\omega_2,\end{aligned}$$

F_t is a symmetry for all t's. ■

Definition 2.11 *By a* ***conservation-law*** *for the Jacobi PDE system* $\Pi = \langle \omega_1, \omega_2 \rangle$*, we mean a differential 1-form* $\theta \in \Omega^1(M)$*, such that:*

$$d\theta = a\omega_1 + b\omega_2,$$

for some $a, b \in C^\infty(M)$*.*

Let θ be a conservation law, and let L be a solution of Π. Assume that $\mathcal{D} \subset L$ is a domain in L, and $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$.

Then:

$$d\theta|_L = 0.$$

Due to Stokes theorem, we know that:

$$\int_{\mathcal{D}} d\theta = \int_{\partial\mathcal{D}} \theta,$$

and so:

$$\int_{\partial\mathcal{D}} \theta = 0.$$

2.2 Examples

2.2.1 Cauchy-Riemann system

In this example we will take the Cauchy-Riemann system, and find its corresponding forms by the method described in section (2.1) :

$$\begin{aligned} \frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2} &= 0 \Longleftrightarrow \omega_1 = dx_1 \wedge du_1 + dx_2 \wedge du_2, \\ \frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2} &= 0 \Longleftrightarrow \omega_2 = du_1 \wedge dx_2 - du_2 \wedge dx_1. \end{aligned}$$

Due to definition (2.11) ,we see that $\theta = u_1 dx_1 + u_2 dx_2$ and $\theta' = u_1 dx_2 + u_2 du_1$ are conservation laws for the Cauchy-Riemann system, since $d\theta = \omega_1$ and $d\theta' = \omega_2$.

Among the simplest examples of symmetries $F : M \longrightarrow M$ of the Cauchy-Riemann system, we have translations:

$$(x_1, x_2, u_1, u_1) \overset{F(a,b,c,d)}{\longmapsto} (x_1 + a, x_2 + b, u_1 + c, u_1 + d) .$$

One can easily see that:

$$\begin{aligned} \Lambda^2 F^* (\omega_1) &= d(u_1 + c) \wedge d(x_1 + a) + d(u_2 + b) \wedge d(x_2 + b) = \omega_1, \\ \Lambda^2 F^* (\omega_2) &= d(u_1 + c) \wedge d(x_2 + b) - d(u_2 + b) \wedge d(x_1 + a) = \omega_2. \end{aligned}$$

Note that ω_1 is the standard symplectic structure on a $\mathbb{R}^4$.

The Hamiltonian vector field X_H is derived by:

$$i_{X_H} \omega_1 = dH.$$

Hence:

$$X_H = \frac{\partial H}{\partial u_1}\frac{\partial}{\partial x_1} + \frac{\partial H}{\partial u_2}\frac{\partial}{\partial x_2} - \frac{\partial H}{\partial x_1}\frac{\partial}{\partial u_1} - \frac{\partial H}{\partial x_2}\frac{\partial}{\partial u_2}.$$

Theorem 2.12 *The Hamiltonian vector field X_H is an infinitesimal symmetry for the*

Cauchy-Riemann system if and only if:

$$\begin{aligned} \frac{\partial^2 H}{\partial x_1 \partial x_1} + \frac{\partial^2 H}{\partial x_2 \partial x_2} &= 0, \\ \frac{\partial^2 H}{\partial u_2 \partial u_2} + \frac{\partial^2 H}{\partial u_1 \partial u_1} &= 0, \\ \frac{\partial^2 H}{\partial u_2 \partial x_1} + \frac{\partial^2 H}{\partial x_2 \partial u_1} &= -\frac{\partial^2 H}{\partial u_1 \partial x_2} - \frac{\partial^2 H}{\partial x_1 \partial u_2}, \\ \frac{\partial^2 H}{\partial x_2 \partial u_2} - \frac{\partial^2 H}{\partial u_1 \partial x_1} &= -\frac{\partial^2 H}{\partial u_2 \partial x_2} + \frac{\partial^2 H}{\partial x_1 \partial u_1}. \end{aligned} \tag{2.2}$$

Proof. Its well known that the Hamiltonian vector fields preserve the standard symplectic structure, that is:

$$L_{X_H}\omega_1 = 0.$$

Note that:

$$i_{X_H} du_i = -\frac{\partial H}{\partial x_i} \text{ and } i_{X_H} dx_i = \frac{\partial H}{\partial u_i}.$$

Let us calculate:

$$\begin{aligned} L_{X_H}\omega_2 &= \\ &= L_{X_H}\left(du_1 \wedge dx_2 - du_2 \wedge dx_1\right) \\ &= L_{X_H}\left(du_1\right) \wedge dx_2 + du_1 \wedge L_{X_H}\left(dx_2\right) - L_{X_H}\left(du_2\right) \wedge dx_1 - du_2 \wedge L_{X_H}\left(dx_1\right) \\ &= -d\left(\frac{\partial H}{\partial x_1}\right) \wedge dx_2 + du_1 \wedge d\left(\frac{\partial H}{\partial u_2}\right) + d\left(\frac{\partial H}{\partial x_2}\right) \wedge dx_1 - du_2 \wedge d\left(\frac{\partial H}{\partial u_1}\right). \end{aligned}$$

Hence, if we write:

$$L_{X_H}\omega_2 = a\omega_1 + b\omega_2,$$

we find that $a = \left(\frac{\partial^2 H}{\partial u_2 \partial x_1} + \frac{\partial^2 H}{\partial x_2 \partial u_1}\right)$ and $b = \left(\frac{\partial^2 H}{\partial x_2 \partial u_2} - \frac{\partial^2 H}{\partial u_1 \partial x_1}\right)$, and conditions (2.2) are satisfied. ■

2.2.2 The symplectic Monge-Ampère equations

The symplectic Monge-Ampère equations have the following form:

$$\hat{a}+b\frac{\partial^2\varphi}{\partial x_1^2}-d\frac{\partial^2\varphi}{\partial x_2^2}-c\frac{\partial^2\varphi}{\partial x_2\partial x_1}+e\frac{\partial^2\varphi}{\partial x_1\partial x_2}+\check{a}\frac{\partial\varphi}{\partial x_1}+\tilde{a}\frac{\partial\varphi}{\partial x_2}+f\left(\frac{\partial^2\varphi}{\partial x_1^2}\frac{\partial^2\varphi}{\partial x_2^2}-\frac{\partial^2\varphi}{\partial x_2\partial x_1}\frac{\partial^2\varphi}{\partial x_1\partial x_2}\right)=0, \tag{i}$$

where $\hat{a},..,f$ are smooth functions of x and $\frac{\partial\varphi}{\partial x}$.

We reduce the symplectic Monge-Ampère equation into a Jacobi PDE system, by the following substitution: $h_1 = \frac{\partial\varphi}{\partial x_1}$, $h_2 = \frac{\partial\varphi}{\partial x_2}$ and $\frac{\partial h_1}{\partial x_2} = \frac{\partial h_2}{\partial x_1}$ as the compatibility condition.

After the substitution, we get the Jacobi system:

$$\begin{aligned} \frac{\partial h_2}{\partial x_1}-\frac{\partial h_1}{\partial x_2} &= 0, \\ a+b\frac{\partial h_1}{\partial x_1}-c\frac{\partial h_1}{\partial x_2}-d\frac{\partial h_2}{\partial x_2}+e\frac{\partial h_2}{\partial x_1}+f\left(\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2}-\frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right) &= 0, \end{aligned} \tag{ii}$$

where $a=\hat{a}+\check{a}h_1+\tilde{a}h_2$ and $a,..,b$ are smooth functions of x, h.

The corresponding 2-forms are:

$$\begin{aligned} \omega_1 &= dx_1\wedge du_1+dx_2\wedge du_2, \\ \omega_2 &= a(x,u)dx_1\wedge dx_2 + b(x,u)du_1\wedge dx_2 \\ &\quad +c(x,u)du_1\wedge dx_1+d(x,u)du_2\wedge dx_1 \\ &\quad +e(x,u)du_2\wedge dx_2 + f(x,u)du_1\wedge du_2 \ . \end{aligned}$$

Theorem 2.13 *If we have a Jacobi PDE system with a conservation law $\theta\in\Omega^1(M)$, such that $d\theta$ is a non-degenerated 2-form , then locally it can be written as the symplectic Monge-Ampère equation (i).*

Proof. The non-degenerated 2-form $d\theta$, determines a symplectic structure on M. Due to Darboux theorem, locally there exists a canonical coordinate system for $d\theta$, say

(x_1, x_2, u_1, u_2) , such that:

$$d\theta = dx_1 \wedge du_1 + dx_2 \wedge du_2.$$

Let ω' be a 2-form, such that $\langle \omega', d\theta \rangle$ is a local basis for Π. Then ω' has the same form as ω_2 in the example above. Hence the Jacobi PDE system $(\Delta_{\omega'}(h) = 0, \Delta_{d\theta}(h) = 0)$, can be written as the symplectic Monge-Ampère equation (i). ■

Chapter 3

Classification of Jacobi planes

In this chapter we will classify Jacobi planes $\Pi \subset \Lambda^2(V^*)$, $\dim V = 4$ with respect to the group $GL(V)$ of linear transformations of V.

3.1 Symmetric bilinear form on the Jacobi planes

Let $\Omega \in \Lambda^4(V^*)$ be a volume form.

Define a symmetric bilinear form q on Π by:

$$\begin{array}{rcl} q : \Pi \otimes \Pi & \longrightarrow & \mathbb{R}, \\ (\alpha, \beta) & \longmapsto & \alpha \wedge \beta = q(\alpha, \beta)\Omega, \end{array}$$

where $\alpha, \beta \in \Pi$.

Let us find a coordinate expression for q.

Assume that e_1, e_2, e_3, e_4 is a basis in V, and let us fix the volume form $\Omega = e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*$.

Then the quadratic form q is represented by the matrix:

$$Q = \begin{bmatrix} q(\omega_1, \omega_1) & q(\omega_1, \omega_2) \\ q(\omega_2, \omega_1) & q(\omega_2, \omega_2) \end{bmatrix}.$$

To find Q, we will assume that ω_i are given as:

$\omega_i = a_i e_1^* \wedge e_2^* + b_i e_3^* \wedge e_2^* + c_i e_3^* \wedge e_1^* + d_i e_4^* \wedge e_1^* + e_i e_4^* \wedge e_2^* + f_i e_3^* \wedge e_4^*.$

Thus,

$$Q = \begin{bmatrix} 2f_1a_1 - 2e_1c_1 + 2b_1d_1 & f_1a_2 + d_1b_2 - e_1c_2 + b_1d_2 - c_1e_2 + a_1f_2 \\ f_1a_2 + d_1b_2 - e_1c_2 + b_1d_2 - c_1e_2 + a_1f_2 & 2f_2a_2 - 2e_2c_2 + 2b_2d_2 \end{bmatrix}.$$

Denote $2f_1a_1 - 2e_1c_1 + 2b_1d_1$ by $\mathcal{A}$, $f_1a_2 + d_1b_2 - e_1c_2 + b_1d_2 - c_1e_2 + a_1f_2$ by $\mathcal{B}$, $2f_2a_2 - 2e_2c_2 + 2b_2d_2$ by $\mathcal{C}$ and $\det Q$ by $\mathcal{K}$.

Note that if we change Ω to $\nu\Omega$, where $\nu \in \mathbb{R}$, then Q changes to $\nu^{-1}Q$. This means that we know Q up to multiplier. We also notice that if we change basis in Π, and let P be the transition matrix, then Q will change the following way:

$$\begin{aligned} Q &\longrightarrow P^T Q P, \text{ then} \\ \det Q &\longrightarrow \det Q \left(\det P\right)^2. \end{aligned}$$

Therefore, $\operatorname{sign}\det Q$ is an invariant of Jacobi planes with respect to $GL(V)$.

Denote $\operatorname{sign}\det Q$ by $\varepsilon(\Pi)$.

3.2 Types of Jacobi planes

3.2.1 Elliptic Jacobi planes

Definition 3.1 *We say that* $\Pi \subset \Lambda^2(V^*)$ *is an* ***elliptic plane*** *if* $q|_\Pi$ *is a* ***non-degenerated determined form.***

Definition 3.2 *Let* Π *be a Jacobi PDE system. We will say that a Jacobi PDE system is elliptic at the point* x*, or simply that* x *is an elliptic point, if the Jacobi plane* $\Pi(x) = \langle \omega_{1,x}, \omega_{2,x} \rangle$ *is* ***elliptic.***

Proposition 3.3 *Let $q|_{\Pi}$ be a non-degenerated determined form, then:*

$$\operatorname{sign}\det Q > 0.$$

Proof. Due to Sylvester's theorem, we may assume that the matrix of the bilinear form q is diagonal, say:

$$Q = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}.$$

Then q is a non-degenerated determined form, if and only if:

$$a_{11}a_{22} > 0.$$

Hence,

$$\operatorname{sign}\det Q > 0.$$

■

Theorem 3.4 *The following two statements are equivalent:*

1) $\Pi = \langle \omega_1, \omega_2 \rangle \subset \Lambda^2(V^*)$ *is an* ***elliptic plane.***

2) There exist a basis $\langle \theta_1, \theta_2 \rangle$ *on* Π, *such that:*

$$\theta_1 \wedge \theta_2 = \theta_2 \wedge \theta_1 = 0 \text{ and } \theta_1 \wedge \theta_1 = \theta_2 \wedge \theta_2 \neq 0.$$

Proof. Let $\{\omega_1, \omega_2\}$ be a basis on Π, and we use the notation from Section (3.1). Note that $\mathcal{A} \neq 0$ and $\mathcal{K} \neq 0$.

Take:

$$\theta_1 = \tfrac{1}{\sqrt{\mathcal{K}}}\omega_1,$$
$$\theta_2 = -\tfrac{\mathcal{B}}{\mathcal{K}}\omega_1 + \tfrac{\mathcal{A}}{\mathcal{K}}\omega_2,$$

in this basis we get that:

$$Q = \begin{bmatrix} \frac{\mathcal{A}}{\mathcal{K}} & 0 \\ 0 & \frac{\mathcal{A}}{\mathcal{K}} \end{bmatrix}.$$

Therefore, in the basis $\{\theta_1, \theta_2\}$ on Π, we have the following relations:

$$\theta_1 \wedge \theta_2 = \theta_2 \wedge \theta_1 = 0 \text{ and } \theta_1 \wedge \theta_1 = \theta_2 \wedge \theta_2.$$

■

3.2.2 Hyperbolic Jacobi planes

Definition 3.5 *We say that $\Pi \subset \Lambda^2(V^*)$ is a* ***hyperbolic plane*** *if $q|_\Pi$ is a* ***non-degenerated sign undetermined form****.*

Definition 3.6 *Let Π be a Jacobi PDE system. We will say that a Jacobi PDE system is hyperbolic at the point x, or simply that x is a hyperbolic point, if the Jacobi plane $\Pi(x) = \langle \omega_{1,x}, \omega_{2,x} \rangle$ is* ***hyperbolic****.*

In a similar way as for proposition (3.3) and theorem (3.8), we obtain the following:

Proposition 3.7 *Let $q|_\Pi$ be a non-degenerated sign undetermined form, then:*

$$\operatorname{sign} \det Q < 0.$$

Theorem 3.8 *The following two statements are equivalent:*

1) $\Pi = \langle \omega_1, \omega_2 \rangle \subset \Lambda^2(V^*)$ is a ***hyperbolic plane.***

2) There exist a basis $\langle \theta_1, \theta_2 \rangle$ on Π, such that:

$$\theta_1 \wedge \theta_2 = \theta_2 \wedge \theta_1 = 0 \text{ and } \theta_1 \wedge \theta_1 = -\theta_2 \wedge \theta_2 \neq 0.$$

Remark 3.9 *For both the elliptic and the hyperbolic plane we may do the following basis change:*

$$\begin{aligned}\theta_1 &= \tfrac{1}{\sqrt{|\mathcal{K}|}}\omega_1 \\ \theta_2 &= -\tfrac{\mathcal{B}}{\mathcal{K}}\omega_1 + \tfrac{\mathcal{A}}{\mathcal{K}}\omega_2\end{aligned} .$$

Then we get:

$$\begin{aligned}\theta_1 \wedge \theta_2 &= 0 \ , \\ \theta_1 \wedge \theta_1 \,\mathrm{sign}(\mathcal{K}) &= \theta_2 \wedge \theta_2.\end{aligned} \tag{3.1}$$

From now on we will refer to such a basis as an ***orthogonal basis.***

3.2.3 Parabolic Jacobi planes

Definition 3.10 *We say that* $\Pi \subset \Lambda^2(V^*)$ *is a* ***parabolic plane*** *if* $q|_\Pi$ *is a* ***degenerated non-zero form.***

Definition 3.11 *We will say that a Jacobi PDE system is parabolic at the point* x*, or simply that* x *is a parabolic point, if the Jacobi plane* $\Pi(x) = \langle \omega_{1,x}, \omega_{2,x} \rangle$ *is* ***parabolic.***

With a similar analysis as above one can derive the following:

Proposition 3.12 *Let* $q|_\Pi$ *be a degenerated non-zero form, then:*

$$\mathrm{sign}\det Q = 0.$$

Theorem 3.13 *The following two statements are equivalent:*

1) $\Pi = \langle \omega_1, \omega_2 \rangle \subset \Lambda^2(V^*)$ *is a* ***parabolic plane.***

2) There exist a basis $\langle \theta_1, \theta_2 \rangle$ *on* Π*, such that:*

$$\theta_1 \wedge \theta_2 = \theta_2 \wedge \theta_1 = 0 \ \textit{and}\ \theta_1 \wedge \theta_1 = 0 \ \textit{and}\ \theta_2 \wedge \theta_2 \neq 0.$$

3.2.4 Degenerated Jacobi planes

Definition 3.14 *We say that* $\Pi \subset \Lambda^2(V^*)$ *is a* ***degenerated plane*** *if* $q|_{\Pi} = 0$.

Definition 3.15 *We will say that a Jacobi PDE system is degenerated at the point* x*, or simply that* x *is a degenerated point, if the Jacobi plane* $\Pi(x) = \langle \omega_{1,x}, \omega_{2,x} \rangle$ *is* ***degenerated****.*

3.3 Classification of elliptic Jacobi planes

In this section, we will classify Jacobi planes with respect to $GL(V)$.

Let (V, Ω) be a 4-dimensional symplectic vector space, where Ω is the symplectic structure on V.

Definition 3.16 *For any* $\theta \in \Lambda^2(V^*)$*, we define the* ***Pfaffian,*** $Pf(\theta) \in \mathbb{R}$*, of the form* θ *in the following way:*

$$\theta \wedge \theta = Pf(\theta)\, \Omega \wedge \Omega.$$

We call θ an **effective** form if:

$$\theta \wedge \Omega = 0.$$

Theorem 3.17 [*LRC*] : *Any effective non-degenerated 2-form* θ*, on the four dimensional symplectic vector space* V*, may be transformed, by means of symplectic transformations, to one of the following:*

elliptic type:

$$\theta = \lambda\left(e_1^* \wedge f_2^* - e_2^* \wedge f_1^*\right), \ \lambda = \sqrt{Pf(\theta)} \text{ and } Pf(\theta) > 0,$$

hyperbolic type:

$$\theta = \lambda\left(e_1^* \wedge f_1^* - e_2^* \wedge f_2^*\right), \ \lambda = \sqrt{-Pf(\theta)} \text{ and } Pf(\theta) < 0,$$

parabolic type:

$$\theta = e_1^* \wedge f_2^*, \text{ and } Pf(\theta) = 0.$$

Let Π be an elliptic plane.

Then due to theorem (3.4), there exists a basis $\{\theta_1, \theta_2\}$ on Π, such that:

$$\theta_1 \wedge \theta_2 = 0,$$

and:

$$\theta_1 \wedge \theta_1 = \theta_2 \wedge \theta_2 \neq 0.$$

Since both θ_1 and θ_2 are non-degenerated, we may choose θ_1 to be a symplectic form on V, and θ_2 to be an effective form, and $Pf(\theta_2) = 1$.

Denote θ_1 by Ω and θ_2 by θ.

Due to theorem (3.17), there exist a basis $\{e_1, e_2, f_1, f_2\}$ on V, such that:

$$\begin{aligned} \theta &= e_1^* \wedge f_2^* - e_2^* \wedge f_1^*, \\ \Omega &= e_1^* \wedge f_1^* + e_2^* \wedge f_2^*, \end{aligned}$$

since $Pf(\theta) = 1$.

Let us call the basis $\{e_1, e_2, f_1, f_2\}$, the canonical basis for V.

We call the forms θ and Ω for the **normal forms** for Π.

Let Π and Π' be two elliptic Jacobi planes, then there is a canonical basis $\{e_1, e_2, f_1, f_2\}$ for Π, and a canonical basis $\{e_1', e_2', f_1', f_2'\}$ for Π'.

Thus, the linear operator $T : V \longrightarrow V$, which acts like:

$$\begin{aligned} e_1 &\longmapsto e_1', \\ e_2 &\longmapsto e_2', \\ f_1 &\longmapsto f_1', \\ f_2 &\longmapsto f_2', \end{aligned}$$

transforms Π to Π'.

The results above give rise to the following theorem:

Theorem 3.18 *1.*

2. *Any two elliptic Jacobi planes* $\Pi, \Pi' \subset \Lambda^2(V^*)$, *are equivalent with respect to* $GL(V)$.

3. *For any elliptic Jacobi plane* Π, *there exist a canonical basis* $\{e_1, e_2, f_1, f_2\}$ *on* V, *such that:*

$$\begin{aligned} \theta &= e_1^* \wedge f_2^* - e_2^* \wedge f_1^*, \\ \Omega &= e_1^* \wedge f_1^* + e_2^* \wedge f_2^*, \end{aligned}$$

where θ *and* Ω *are the* ***normal forms*** *for* Π.

3.4 Classification of hyperbolic Jacobi planes

With a similar analysis as in section (3.3), we get:

Theorem 3.19 *1. Any two hyperbolic Jacobi planes* $\Pi, \Pi' \subset \Lambda^2(V^*)$, *are equivalent with respect to* $GL(V)$.

2. *For any hyperbolic Jacobi plane* Π, *there exist a canonical basis* $\{e_1, e_2, f_1, f_2\}$ *on* V, *such that:*

$$\begin{aligned} \theta &= e_1^* \wedge f_1^* - e_2^* \wedge f_2^*, \\ \Omega &= e_1^* \wedge f_1^* + e_2^* \wedge f_2^*, \end{aligned}$$

where θ *and* Ω *are the* ***normal forms*** *for* Π.

3.5 Classification of parabolic Jacobi planes

With a similar analysis as in section (3.3), we get:

Theorem 3.20 *1. Any two parabolic Jacobi planes* $\Pi, \Pi' \subset \Lambda^2(V^*)$, *are equivalent with respect to* $GL(V)$.

2. For any parabolic Jacobi plane Π, *there exist a canonical basis* $\{e_1, e_2, f_1, f_2\}$ *on* V, *such that:*

$$\begin{aligned} \theta &= e_1^* \wedge f_2^*, \\ \Omega &= e_1^* \wedge f_1^* + e_2^* \wedge f_2^*, \end{aligned}$$

where θ *and* Ω *are the* ***normal forms*** *for* Π.

3.6 Classification of degenerated Jacobi planes

One can easily show with some linear algebra, that the following holds:

Theorem 3.21 *1. Any two degenerated Jacobi planes* $\Pi, \Pi' \subset \Lambda^2(V^*)$, *are equivalent with respect to* $GL(V)$.

2. For any degenerated Jacobi plane Π, *there exist a canonical basis* $\{e_1, e_2, f_1, f_2\}$ *on* V, *such that:*

$$\begin{aligned} \omega_1 &= e_1^* \wedge f_2^*, \\ \omega_2 &= e_1^* \wedge f_1^*, \end{aligned}$$

where ω_1 *and* ω_2 *are the* ***normal forms*** *for* Π.

3.6.1 Invariants of elliptic and hyperbolic Jacobi PDE systems

As a result of the classifications, we obtain the following theorems.

Theorem 3.22 *1.* $\varepsilon(\Pi)$ *is the only invariant for non-degenerated Jacobi planes with respect to* $GL(V)$*.*

2. $\varepsilon(\Pi) = \mathrm{sign}(2e_1e_2c_1c_2 + 2e_1b_1c_2d_2 - 4e_1b_2c_1d_2 + 2e_1b_2c_2d_1 + 2e_2b_1c_1d_2 - 4e_2b_1c_2d_1 + 2e_2b_2c_1d_1 + 2e_1a_1c_2f_2 - 4e_1a_2c_1f_2 + 2e_1a_2c_2f_1 + 2e_2a_1c_1f_2 - 4e_2a_1c_2f_1 + 2e_2a_2c_1f_1 + 2b_1b_2d_1d_2 - 2a_1b_1d_2f_2 - 2a_1b_2d_1f_2 + 4a_1b_2d_2f_1 + 4a_2b_1d_1f_2 - 2a_2b_1d_2f_1 - 2a_2b_2d_1f_1 + 2a_1a_2f_1f_2 - e_1^2c_2^2 - e_2^2c_1^2 - b_1^2d_2^2 - b_2^2d_1^2 - a_1^2f_2^2 - a_2^2f_1^2)$.

3.6.2 Application to symmetries

Since we found the orthogonal basis $\{\theta_1, \theta_2\}$, we will review the definitions for symmetries.

Assume that $\Delta_{\theta_1}(h) = 0, \Delta_{\theta_2}(h) = 0$ is a Jacobi PDE system, and that $\{\theta_1, \theta_2\}$ is an orthogonal basis. Further on we assume that all of the Jacobi planes $\Pi = \langle\theta_1, \theta_2\rangle$, are either elliptic or hyperbolic.

Proposition 3.23 *Let* $F : M \longrightarrow M$ *be a symmetry for the Jacobi PDE system* Π*, and*

$$\begin{aligned} \Lambda^2 F^*(\theta_1) &= a_{11}\theta_1 + a_{12}\theta_2, \\ \Lambda^2 F^*(\theta_2) &= a_{21}\theta_1 + a_{22}\theta_2, \end{aligned}$$

for some $a_{ij} \in C^\infty(M)$.

Then:

$$a_{21}^2 + \varepsilon(\Pi)\, a_{22}^2 = \varepsilon(\Pi)\, a_{11}^2 + a_{12}^2 \text{ and } a_{11}a_{21} + \varepsilon(\Pi)\, a_{12}a_{22} = 0.$$

Proof. Since $\{\theta_1, \theta_2\}$ is an orthogonal basis, we know that:

$$\theta_1 \wedge \theta_2 = \theta_2 \wedge \theta_1 = 0 \text{ and } \theta_1 \wedge \theta_1 \varepsilon (\Pi) = \theta_2 \wedge \theta_2.$$

Further on we notice that:

$$\Lambda^4 F^* (\theta_2 \wedge \theta_2) = \varepsilon (\Pi) \Lambda^4 F^* (\theta_1 \wedge \theta_1) .$$

Hence:

$$a_{21}^2 \theta_1 \wedge \theta_1 + \varepsilon (\Pi) a_{22}^2 \theta_1 \wedge \theta_1 = \varepsilon (\Pi) \left(a_{11}^2 \theta_1 \wedge \theta_1 + \varepsilon (\Pi) a_{12}^2 \theta_1 \wedge \theta_1 \right) ,$$

and therefore:

$$a_{21}^2 + \varepsilon (\Pi) a_{22}^2 = \varepsilon (\Pi) a_{11}^2 + a_{12}^2.$$

Since $\theta_1 \wedge \theta_2 = 0$, we get:

$$\Lambda^4 F^* (\theta_1 \wedge \theta_2) = 0,$$

and so:

$$a_{11} a_{21} \theta_1 \wedge \theta_1 + a_{12} a_{22} \theta_2 \wedge \theta_2 = 0,$$

and:

$$a_{11} a_{21} + \varepsilon (\Pi) a_{12} a_{22} = 0.$$

■

From the proposition above, we derive the following two corollaries:

Corollary 3.24 *If the Jacobi PDE system* Π *is elliptic, then the matrix* $||a_{ij}||$ *is a conformal orthogonal matrix with respect to the elliptic (standard) metric.*

Corollary 3.25 *If the Jacobi PDE system* Π *is hyperbolic, then the matrix* $||a_{ij}||$ *is a conformal orthogonal matrix with respect to the hyperbolic metric.*

Proposition 3.26 *Let $X \in \mathcal{D}(M)$ be an* ***infinitesimal symmetry*** *of the Jacobi PDE system* Π, *and:*

$$\begin{aligned} L_X(\theta_1) &= a_{11}\theta_1 + a_{12}\theta_2, \\ L_X(\theta_2) &= a_{21}\theta_1 + a_{22}\theta_2, \end{aligned}$$

for some $a_{ij} \in C^\infty(M)$.

Then:

$$\begin{aligned} a_{11} &= a_{22} \\ a_{12} &= -\varepsilon(\Pi)\, a_{21}. \end{aligned}$$

Proof. Since $\{\theta_1, \theta_2\}$ is an orthogonal basis, we know that:

$$\theta_1 \wedge \theta_2 = \theta_2 \wedge \theta_1 = 0 \text{ and } \theta_1 \wedge \theta_1 \varepsilon(\Pi) = \theta_2 \wedge \theta_2.$$

Due to the Leibniz-rule, we get:

$$L_X(\theta_1 \wedge \theta_2) = L_X(\theta_1) \wedge \theta_2 + \theta_1 \wedge L_X(\theta_2) = 0.$$

Therefore:

$$0 = a_{12}\theta_2 \wedge \theta_2 + a_{21}\theta_1 \wedge \theta_1,$$

and so:

$$-\varepsilon(\Pi)\, a_{12} = a_{21}.$$

By the linearity of the Lie derivative, we get that:

$$L_X(\theta_2 \wedge \theta_2) = \varepsilon(\Pi)\, L_X(\theta_1 \wedge \theta_1).$$

Hence:

$$a_{22}\theta_2 \wedge \theta_2 = \varepsilon(\Pi)\, a_{11}\theta_1 \wedge \theta_1,$$

and so:

$$a_{22} = a_{11}.$$

■

3.7 Examples

To show how this works, we will take a few PDE's, and classify them according to the definitions above.

3.7.1 Cauchy-Riemann equations

Take the Cauchy-Riemann system:

$$\begin{aligned} \frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2} &= 0, \\ \frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2} &= 0. \end{aligned}$$

Comparing this with the Jacobi PDE system $(*)$, we get that: $c_1 = 1, e_1 = 1, b_2 = 1$ and $d_2 = -1$.

We calculate :

$$\varepsilon(\Pi) = \operatorname{sign}(-4e_1c_1b_2d_2) = \operatorname{sign}(4) > 0.$$

This means that the system is elliptic at any point.

3.7.2 A hyperbolic PDE system

Let us consider the following PDE system, and analyze it:

$$\begin{aligned} h_2 \frac{\partial h_1}{\partial x_1} - \frac{\partial h_1}{\partial x_2} + \frac{1}{h_1 - h_2} &= 0, \\ h_1 \frac{\partial h_2}{\partial x_1} - \frac{\partial h_2}{\partial x_2} + \frac{1}{h_2 - h_1} &= 0, \end{aligned}$$

where h_1 and h_2 are the unknown smooth functions of the independent variables x_1 and x_2. (This system was presented to me by Professor E. Ferapontov during a conference in Opava in the Autumn 2001.)

$$\varepsilon\left(\Pi\right) = \operatorname{sign}\left(-\left(u_2 - u_1\right)^2\right) < 0.$$

Since $u_2 - u_1 \neq 0$, we conclude that the system is hyperbolic at any point.

3.7.3 The symplectic Monge-Ampère equations

Now we shall analyze the symplectic Monge-Ampère equations (i) , given in example(2.2.2) . With the same substitution as in example(2.2.2) , we obtain the system:

$$\begin{aligned} 0 &= \frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2}, \\ 0 &= a + b\frac{\partial h_1}{\partial x_1} - c\frac{\partial h_1}{\partial x_2} - d\frac{\partial h_2}{\partial x_2} + e\frac{\partial h_2}{\partial x_1} + f\left(\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right), \end{aligned}$$

where $a = \hat{a} + \check{a}h_1 + \tilde{a}h_2$, and find:

$$\varepsilon\left(\Pi\right) = \operatorname{sign}\left(-4\left(af + bd\right) - \left(c - e\right)^2\right).$$

Summing up the results above, we get the following proposition:

Proposition 3.27 *The symplectic Monge-Ampère equations:*

$$\hat{a}+b\frac{\partial^2\varphi}{\partial x_1^2}-d\frac{\partial^2\varphi}{\partial x_2^2}-c\frac{\partial^2\varphi}{\partial x_2\partial x_1}+e\frac{\partial^2\varphi}{\partial x_1\partial x_2}+\check{a}\frac{\partial\varphi}{\partial x_1}+\tilde{a}\frac{\partial\varphi}{\partial x_2}+f\left(\frac{\partial^2\varphi}{\partial x_1^2}\frac{\partial^2\varphi}{\partial x_2^2}-\frac{\partial^2\varphi}{\partial x_2\partial x_1}\frac{\partial^2\varphi}{\partial x_1\partial x_2}\right)=0,$$

have the following classifications:

- elliptic, whenever $-4\left(\left(\hat{a}+\check{a}h_1+\tilde{a}h_2\right)f+bd\right)>\left(c-e\right)^2$,
- hyperbolic, whenever $-4\left(\left(\hat{a}+\check{a}h_1+\tilde{a}h_2\right)f+bd\right)<\left(c-e\right)^2$,
- parabolic, whenever $-4\left(\left(\hat{a}+\check{a}h_1+\tilde{a}h_2\right)f+bd\right)=\left(c-e\right)^2$.

Second order quasi-linear equation

As the last example, we will analyze a second order quasi-linear equation with φ as the unknown function. The equation is given by:

$$a\left(x_1,x_2\right)\frac{\partial^2\varphi}{\partial x_1^2}+2b\left(x_1,x_2\right)\frac{\partial^2\varphi}{\partial x_1\partial x_2}+c\left(x_1,x_2\right)\frac{\partial^2\varphi}{\partial x_2^2}=e\left(x_1,x_2,\frac{\partial\varphi}{\partial x_1},\frac{\partial\varphi}{\partial x_2}\right),$$

where a,b,c are smooth functions of the independent variables x_1,x_2, and e is a smooth function of $x_1,x_2,\frac{\partial\varphi}{\partial x_1},\frac{\partial\varphi}{\partial x_2}$. This is a special case of the symplectic Monge-Ampère equation classified above, and therefore we derive the following corollary:

Corollary 3.28 *The second order quasi-linear equations given above, have the classifications:*

- elliptic, if $ac-b^2>0$,
- hyperbolic, if $ac-b^2<0$,
- parabolic, if $ac-b^2=0$.

Chapter 4

Operator representation of Jacobi planes

In this chapter we will find another description of the Jacobi planes. From now on we will only consider elliptic and hyperbolic planes.

4.1 Preliminaries

Let V be a vector space, and $\omega \in \Lambda^2(V^*)$ a 2-form. Then ω determines a linear operator $\tilde{\omega} : V \longrightarrow V^*$ in the following way:

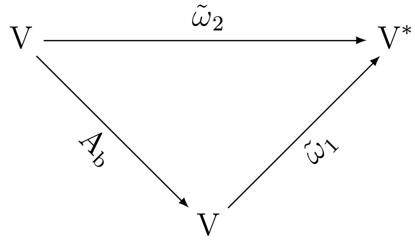
$$X \longmapsto \omega(X, *).$$

If ω is a non-degenerated form, then $\tilde{\omega}$ is a linear isomorphism.

Let $\{\omega_1, \omega_2\}$, $\omega_i \in \Lambda^2(V^*)$ be an oriented basis, and ω_1 be a non-degenerated 2-form.

Denote the oriented basis $\{\omega_1, \omega_2\}$ by b, and the basis $\{\omega_2, \omega_1\}$ with the opposite orientation is denoted by b^0.

We define the operator $A_b : V \longrightarrow V$, by:



Due to the diagram, it is clear that:

$$A_b = \tilde{\omega}_1^{-1} \circ \tilde{\omega}_2.$$

Alternatively, $A_b : V \longrightarrow V$ is an operator such that:

$$\omega_2(X, Y) = \omega_1(A_b X, Y),$$

holds for all $X, Y \in V$.

Note that if ω_1 and ω_2 both are non-degenerated 2-forms, then:

$$A_{b^0} = A_b^{-1} = \tilde{\omega}_2^{-1} \circ \tilde{\omega}_1. \tag{4.1}$$

4.2 Operator representation for elliptic Jacobi planes

Let $\Pi \subset \Lambda^2 (V^*)$ be an elliptic Jacobi plane.

Hence, we know from previous analysis that there exist an orthogonal basis $\{\theta_1, \theta_2\}$ on Π, and both θ_1 and θ_2 are non-degenerated.

Further on we may choose θ_1 to be a symplectic form on V, and θ_2 to be an effective form, such that:

$$\theta_1 \wedge \theta_2 = \theta_2 \wedge \theta_1 = 0 \text{ and } \theta_1 \wedge \theta_1 = \theta_2 \wedge \theta_2.$$

Let (V, Ω) be a 4-dimensional symplectic vector space, where Ω is the symplectic structure on V.

Assume that $b' = \{\Omega, \theta\}$.

Theorem 4.1 $[Ly1]$: *For any effective non-degenerated 2-form θ on the four dimensional symplectic vector space V, the following holds:*

$$A_{b'}^2 = -Pf(\theta).$$

Hence, for an ordered orthogonal basis $b = \{\theta_1, \theta_2\}$, we get:

$$A_b^2 = -1,$$

if Π is elliptic.

Note that:

$$A_b^{-1} = -A_b.$$

Hence, for the basis b^0 with the opposite orientation, we get that:

$$A_{b^0} = -A_b \text{ and } A_{b^0}^2 = -1.$$

Let $T : V \longrightarrow V$ be a linear operator on the vector space V.

The pair (V, T), is called a **complex structure,** if T satisfies the condition $T^2 = -1$.

Definition 4.2 *We will call a transformation from the orthogonal basis $\left\{\hat{\theta}_1, \hat{\theta}_2\right\}$ to an orthogonal basis $\{\theta_1, \theta_2\}$, an **elliptic similitude**, if there exist t and ϕ, such that:*

$$\begin{aligned} \theta_1 &= t\cos(\phi)\hat{\theta}_1 - t\sin(\phi)\hat{\theta}_2, \\ \theta_2 &= t\sin(\phi)\hat{\theta}_1 + t\cos(\phi)\hat{\theta}_2. \end{aligned}$$

Theorem 4.3 *Let V be a real vector space and $\Pi \subset \Lambda^2(V^*)$ be an oriented Jacobi elliptic* *plane with an ordered orthogonal basis* $b = \{\theta_1, \theta_2\}$ *. The operator* A_b *does not depend on the choice of ordered orthogonal basis in* Π.

Proof. Let $\hat{A}_{\hat{b}}$ be a complex structure to the corresponding orthogonal basis $\left\{\hat{\theta}_1, \hat{\theta}_2\right\}$, and A_b the complex structure corresponding to the orthogonal basis $\{\theta_1, \theta_2\}$, and $\left\langle\hat{\theta}_1, \hat{\theta}_2\right\rangle = \langle\theta_1, \theta_2\rangle = \Pi$. We must then show that if we do an elliptic similitude to the basis $\{\theta_1, \theta_2\}$, the complex structure A_b is equal to the complex structure $\hat{A}_{\hat{b}}$.

The complex structure $A_{b,}$ is defined by:

$$\theta_2(X, Y) = \theta_1(A_b X, Y).$$

Note that:

$$\hat{\theta}_2(X, Y) = \hat{\theta}_1(\hat{A}_{\hat{b}} X, Y) \Longleftrightarrow \hat{\theta}_1(X, Y) = -\hat{\theta}_2(\hat{A}_{\hat{b}} X, Y).$$

Let the elliptic similitude be given by:

$$\begin{aligned} \theta_1 &= t\cos(\phi)\hat{\theta}_1 - t\sin(\phi)\hat{\theta}_2, \\ \theta_2 &= t\sin(\phi)\hat{\theta}_1 + t\cos(\phi)\hat{\theta}_2. \end{aligned}$$

Inserting this into $\theta_2(X, Y) = \theta_1(A_b X, Y)$, we get:

$$\begin{aligned} \sin(\phi)\hat{\theta}_1(X, Y) + \cos(\phi)\hat{\theta}_2(X, Y) &= \cos(\phi)\hat{\theta}_1(A_b X, Y) - \sin(\phi)\hat{\theta}_2(A_b X, Y) \\ \hat{\theta}_1((\sin(\phi) - \cos(\phi)A_b)\, X, Y) &= -\hat{\theta}_2((\cos(\phi) + \sin(\phi)A_b)\, X, Y). \end{aligned}$$

Substitution:

$$X = (\sin(\phi) + \cos(\phi)A_b)\, \hat{X}.$$

Further on we notice that:

$$
\begin{aligned}
(\cos(\phi) + \sin(\phi)A_b)\,(\sin(\phi) + \cos(\phi)A_b) &= A_b \text{ and} \\
(\sin(\phi) - \cos(\phi)A_b)\,(\sin(\phi) + \cos(\phi)A_b) &= 1.
\end{aligned}
$$

Thus, we get:

$$\hat{\theta}_1(\hat{X}, Y) = -\hat{\theta}_2(A_b\hat{X}, Y).$$

What we noticed early on in the proof is that:

$$\hat{\theta}_1(X, Y) = -\hat{\theta}_2(\hat{A}_{\hat{b}}X, Y).$$

Therefore:

$$A_b = \hat{A}_{\hat{b}}.$$

■

Since A_b does not depend on the choice of oriented orthogonal basis in Π, we will denote the operator A_b by A_Π.

Theorem 4.4 *Any oriented elliptic Jacobi plane* Π*, determines a complex structure* (V, A_Π)*.*

4.2.1 Elliptic Jacobi PDE systems represented by smooth fields of operators

Finally, we sum up all of the results in the previous section to make the link from a smooth field of two-dimensional planes $\Pi\,(x)$ in $\Lambda^2(T_x^*M)$, to a smooth field of operators A_x on T_xM. This is done in the theorem below.

A smooth field of endomorphisms $A_x : T_xM \longrightarrow T_xM$ on a manifold M is called an **almost product complex structure** on M, if $A_x^2 = -1$ for all $x \in M$.

Theorem 4.5 *Let Π be an elliptic Jacobi PDE system with fixed orientation of all the Jacobi planes $\Pi(x)$. Then the smooth field of two-dimensional planes $\Pi(x)$ in $\Lambda^2(T_x^*M)$, determines an almost complex structure on M.*

4.3 Operator representation for hyperbolic Jacobi planes

Let $\Pi \subset \Lambda^2(V^*)$ be a hyperbolic Jacobi plane.

Hence we know from previous analysis that there exists an orthogonal basis $\{\theta_1, \theta_2\}$ on Π, and both θ_1 and θ_2 are non-degenerated. Further on we may choose θ_1 to be a symplectic form on V, and θ_2 to be an effective form, such that:

$$\theta_1 \wedge \theta_2 = \theta_2 \wedge \theta_1 = 0 \text{ and } \theta_1 \wedge \theta_1 = -\theta_2 \wedge \theta_2.$$

Then due to theorem (4.1), we get that for an oriented orthogonal basis $b = \{\theta_1, \theta_2\}$:

$$A_b^2 = 1,$$

if Π is hyperbolic.

Let $B : V \longrightarrow V$ be a linear operator on the vector space V.

The pair (V, B), is a **product structure,** if B satisfy the condition $B^2 = 1$.

Definition 4.6 *We will call a transformation from the orthogonal basis $\left\{\hat{\theta}_1, \hat{\theta}_2\right\}$ to an orthogonal basis $\{\theta_1, \theta_2\}$, a **hyperbolic similitude**, if there exist t and ϕ, such that:*

$$\begin{aligned}
\theta_1 &= t\cosh(\phi)\hat{\theta}_1 + t\sinh(\phi)\hat{\theta}_2, \\
\theta_2 &= t\sinh(\phi)\hat{\theta}_1 + t\cosh(\phi)\hat{\theta}_2.
\end{aligned}$$

Theorem 4.7 *Let V be a real vector space, and $\Pi \subset \Lambda^2(V^*)$ be a hyperbolic Jacobi plane* *with a orthogonal basis* $b = \{\theta_1, \theta_2\}$. *The operator* A_b *does not depend on the choice of orthogonal basis in* Π, *and orientation of* Π.

Proof. Let $\hat{A}_{\hat{b}}$ be a product structure corresponding to the oriented orthogonal basis $\left\{\hat{\theta}_1, \hat{\theta}_2\right\}$, and A_b the product structure corresponding to the oriented orthogonal basis $\{\theta_1, \theta_2\}$, and $\left\langle \hat{\theta}_1, \hat{\theta}_2 \right\rangle = \langle \theta_1, \theta_2 \rangle = \Pi$. We will show that if we do a hyperbolic similitude to the basis $\{\theta_1, \theta_2\}$, then A_b is equal $\hat{A}_{\hat{b}}$.

The product structure A_b, is defined by:

$$\theta_2(X, Y) = \theta_1(A_b X, Y).$$

Note that:

$$\hat{\theta}_2(X, Y) = \hat{\theta}_1(A_{\hat{b}} X, Y) \Longleftrightarrow \hat{\theta}_2(A_{\hat{b}} X, Y) = \hat{\theta}_1(X, Y).$$

Let the hyperbolic similitude be given by:

$$\begin{array}{l} \theta_1 = t\cosh(\phi)\hat{\theta}_1 + t\sinh(\phi)\hat{\theta}_2 \\ \theta_2 = t\sinh(\phi)\hat{\theta}_1 + t\cosh(\phi)\hat{\theta}_2 \end{array}.$$

Inserting this into $\theta_2(X, Y) = \theta_1(A_b X, Y)$, we get:

$$\begin{aligned} \left(t\sinh(\phi)\hat{\theta}_1 + t\cosh(\phi)\hat{\theta}_2\right)(X, Y) &= \left(t\cosh(\phi)\hat{\theta}_1 + t\sinh(\phi)\hat{\theta}_2\right)(A_b X, Y) \\ \hat{\theta}_1((\sinh(\phi) - \cosh(\phi)A_b)\, X, Y) &= \hat{\theta}_2((-\cosh(\phi) + \sinh(\phi)A_b)\, X, Y). \end{aligned}$$

Substitution:

$X = (\cosh(\phi) + \sinh(\phi)A_b)\, \hat{X}$, and we notice that:

$$\begin{aligned} (\sinh(\phi) - \cosh(\phi)A_b)(\cosh(\phi) + \sinh(\phi)A_b) &= A_b, \\ (-\cosh(\phi) + \sinh(\phi)A_b)(\cosh(\phi) + \sinh(\phi)A_b) &= 1. \end{aligned}$$

Thus, we get that:

$$\begin{aligned}\hat{\theta}_1(A_b\hat{X}, Y) &= \hat{\theta}_2(\hat{X}, Y),\\ \hat{\theta}_1(A_{\hat{b}}X, Y) &= \hat{\theta}_2(X, Y).\end{aligned}$$

Hence:

$$A_b = A_{\hat{b}}.$$

The operator A_b does not depend on the choice of orientation of Π, since:

$$A_b = A_b^{-1} \text{ and } A_b^{-1} = A_{b^0},$$

therefore:

$$A_{b^0} = A_b.$$

■

Since A_b does not depend on the choice of basis in Π, we will denote the operator A_b by A_Π.

In section (5.1), we will prove that $\dim\ker(1 \pm A_\Pi) = 2$ for hyperbolic Jacobi planes.

This motivates the following definition:

Definition 4.8 *Let $B : V \longrightarrow V$ be a linear operator on the vector space V. The pair (V, B), will be called a* ***symmetric product structure*** *if B satisfy the condition $B^2 = 1$, and* $\dim\ker(1 - B) = \dim\ker(1 + B)$.

Therefore, we sum up the results in this section by the following theorem:

Theorem 4.9 *Any hyperbolic Jacobi plane Π, determines a symmetric product structure (V, A_Π).*

4.3.1 Hyperbolic Jacobi PDE systems represented by smooth fields of operators

Due to the results in the previous section, we make the link from a smooth field of two-dimensional planes $\Pi(x)$ in $\Lambda^2(T_x^*M)$, to a smooth field of operators A_x on T_xM. This is done in the theorem below.

A smooth field of endomorphisms $B_x : T_xM \longrightarrow T_xM$ on a manifold M, is called a **symmetric almost product structure** on M, if $B_x^2 = 1$ for all $x \in M$, and $\dim\ker(1 - B_x) = \dim\ker(1 + B_x)$.

Theorem 4.10 *Let Π be a hyperbolic Jacobi PDE system. Then the smooth field of two-dimensional planes $\Pi(x)$ in $\Lambda^2(T_x^*M)$, determines a symmetric almost product structure $A_x : T_xM \longrightarrow T_xM$ on M.*

4.4 Matrix representation for operators A_Π

Theorem 4.11 *Let Π be a Jacobi PDE system:*

$$\begin{cases} a_1 + b_1\frac{\partial h_1}{\partial x_1} - c_1\frac{\partial h_1}{\partial x_2} - d_1\frac{\partial h_2}{\partial x_2} + e_1\frac{\partial h_2}{\partial x_1} + f_1\left(\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right) = 0 \\ a_2 + b_2\frac{\partial h_1}{\partial x_1} - c_2\frac{\partial h_1}{\partial x_2} - d_2\frac{\partial h_2}{\partial x_2} + e_2\frac{\partial h_2}{\partial x_1} + f_2\left(\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right) = 0 \end{cases},$$

that are either elliptic or hyperbolic. The matrix $||A_x||$ of the operator field A in the standard basis $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right)$, is given as:

$$||A_x|| = \frac{\sqrt{|\mathcal{K}|}}{\mathcal{K}}\begin{bmatrix} \Delta_1 & 2e_2b_1 - 2e_1b_2 & 2b_2f_1 - 2b_1f_2 & 2e_2f_1 - 2e_1f_2 \\ 2c_2d_1 - 2c_1d_2 & \Delta_2 & 2c_1f_2 - 2c_2f_1 & 2d_1f_2 - 2d_2f_1 \\ 2a_2d_1 - 2a_1d_2 & 2e_1a_2 - 2e_2a_1 & \Delta_3 & 2e_2d_1 - 2e_1d_2 \\ 2a_1c_2 - 2a_2c_1 & 2a_1b_2 - 2a_2b_1 & 2b_1c_2 - 2b_2c_1 & \Delta_4 \end{bmatrix}, \quad (**)$$

where:

$$\begin{aligned}
\Delta_1 &= e_2c_1 - e_1c_2 + b_1d_2 - b_2d_1 - a_1f_2 + a_2f_1,\\
\Delta_2 &= e_1c_2 - e_2c_1 - b_1d_2 + b_2d_1 - a_1f_2 + a_2f_1,\\
\Delta_3 &= e_2c_1 - e_1c_2 - b_1d_2 + b_2d_1 + a_1f_2 - a_2f_1,\\
\Delta_4 &= e_1c_2 - e_2c_1 + b_1d_2 - b_2d_1 + a_1f_2 - a_2f_1,\\
\mathcal{K} &= \det Q.
\end{aligned}$$

Proof. Let $\Pi(x) = \langle \omega_{1,x}, \omega_{2,x} \rangle$.

Then:

$$A_x = \tilde{\theta}_{1,x}^{-1} \circ \tilde{\theta}_{2,x},$$

and we find the matrix of the operator A_x denoted by $||A_x||$, if we find $||\tilde{\theta}_{1,x}^{-1}||$ and $||\tilde{\theta}_{2,x}||$, since:

$$||A_x|| = ||\tilde{\theta}_{1,x}^{-1}|| \cdot ||\tilde{\theta}_{2,x}||,$$

where $||\tilde{\theta}_{1,x}^{-1}||$ and $||\tilde{\theta}_{2,x}||$ are the matrix representations of the operators $\tilde{\theta}_{1,x}^{-1}$ and $\tilde{\theta}_{2,x}$.

The easiest way to find, say $||\tilde{\theta}_{2,x}||$, is to use the following relation:

$$\theta_2(X, Y) = X^T ||\tilde{\theta}_{2,x}|| Y.$$

As we know from the orthogonalization, $\theta_{2,x}$ is equal to $-\frac{\mathcal{B}}{\mathcal{K}}\omega_{1,x} + \frac{\mathcal{A}}{\mathcal{K}}\omega_{2,x}$, and we derive the following table:

$\theta_{2,x}(\downarrow, \longrightarrow)$	e_1	e_2	e_3	e_4
e_1	0	$\frac{(\mathcal{A}a_2-\mathcal{B}a_1)}{\mathcal{K}}$	$\frac{-(\mathcal{A}c_2-\mathcal{B}c_1)}{\mathcal{K}}$	$\frac{-(\mathcal{A}d_2-\mathcal{B}d_1)}{\mathcal{K}}$
e_2	$\frac{-(\mathcal{A}a_2-\mathcal{B}a_1)}{\mathcal{K}}$	0	$\frac{-(\mathcal{A}b_2-\mathcal{B}b_1)}{\mathcal{K}}$	$\frac{-(\mathcal{A}e_2-\mathcal{B}e_1)}{\mathcal{K}}$
e_3	$\frac{(\mathcal{A}c_2-\mathcal{B}c_1)}{\mathcal{K}}$	$\frac{(\mathcal{A}b_2-\mathcal{B}b_1)}{\mathcal{K}}$	0	$\frac{(\mathcal{A}f_2-\mathcal{B}f_1)}{\mathcal{K}}$
e_4	$\frac{(\mathcal{A}d_2-\mathcal{B}d_1)}{\mathcal{K}}$	$\frac{(\mathcal{A}e_2-\mathcal{B}e_1)}{\mathcal{K}}$	$\frac{-(\mathcal{A}f_2-\mathcal{B}f_1)}{\mathcal{K}}$	0

.

From this table we can now construct $||\tilde{\theta}_{2,x}||$:

$$||\tilde{\theta}_{2,x}|| = \frac{\mathcal{A}}{\mathcal{K}} \begin{bmatrix} 0 & a_2 & -c_2 & -d_2 \\ -a_2 & 0 & -b_2 & -e_2 \\ c_2 & b_2 & 0 & f_2 \\ d_2 & e_2 & -f_2 & 0 \end{bmatrix} - \frac{\mathcal{B}}{\mathcal{K}} \begin{bmatrix} 0 & a_1 & -c_1 & -d_1 \\ -a_1 & 0 & -b_1 & -e_1 \\ c_1 & b_1 & 0 & f_1 \\ d_1 & e_1 & -f_1 & 0 \end{bmatrix}.$$

With the same procedure as above we are able to find the matrix representation $||\tilde{\theta}_{1,x}||$ of the operator $\tilde{\theta}_{1,x}$.

$$||\tilde{\theta}_{1,x}|| = \frac{1}{\sqrt{|\mathcal{K}|}} \begin{bmatrix} 0 & a_1 & -c_1 & -d_1 \\ -a_1 & 0 & -b_1 & -e_1 \\ c_1 & b_1 & 0 & f_1 \\ d_1 & e_1 & -f_1 & 0 \end{bmatrix}.$$

To make calculations more visible, we do the following substitution:

$$\begin{aligned} \theta_{1,x} &= \hat{a}_1 e_1^* \wedge e_2^* + \hat{b}_1 e_3^* \wedge e_2^* + \hat{c}_1 e_3^* \wedge e_1^* + \hat{d}_1 e_4^* \wedge e_1^* + \hat{e}_1 e_4^* \wedge e_2^* + \hat{f}_1 e_3^* \wedge e_4^*, \\ \theta_{2,x} &= \hat{a}_2 e_1^* \wedge e_2^* + \hat{b}_2 e_3^* \wedge e_2^* + \hat{c}_2 e_3^* \wedge e_1^* + \hat{d}_2 e_4^* \wedge e_1^* + \hat{e}_2 e_4^* \wedge e_2^* + \hat{f}_2 e_3^* \wedge e_4^*, \end{aligned}$$

where $\hat{a}_1 = \frac{a_1}{\sqrt{|\mathcal{K}|}}, .., \hat{f}_1 = \frac{f_1}{\sqrt{|\mathcal{K}|}}$ and $\hat{a}_2 = \frac{(\mathcal{A}a_2 - \mathcal{B}a_1)}{\mathcal{K}}, .., \hat{f}_2 = \frac{(\mathcal{A}f_2 - \mathcal{B}f_1)}{\mathcal{K}}$.
With these substitutions, we get that:

$$||\tilde{\theta}_{1,x}|| = \begin{bmatrix} 0 & \hat{a}_1 & -\hat{c}_1 & -\hat{d}_1 \\ -\hat{a}_1 & 0 & -\hat{b}_1 & -\hat{e}_1 \\ \hat{c}_1 & \hat{b}_1 & 0 & \hat{f}_1 \\ \hat{d}_1 & \hat{e}_1 & -\hat{f}_1 & 0 \end{bmatrix} \text{ and } ||\tilde{\theta}_{2,x}|| = \begin{bmatrix} 0 & \hat{a}_2 & -\hat{c}_2 & -\hat{d}_2 \\ -\hat{a}_2 & 0 & -\hat{b}_2 & -\hat{e}_2 \\ \hat{c}_2 & \hat{b}_2 & 0 & \hat{f}_2 \\ \hat{d}_2 & \hat{e}_2 & -\hat{f}_2 & 0 \end{bmatrix}.$$

Thus, we find the expression for $||A_x||$:

$$||A_x|| = ||\tilde{\theta}^{-1}_{1,x}|| \cdot ||\tilde{\theta}_{2,x}|| =$$

$$\frac{1}{\Delta_0}\begin{bmatrix} \tilde{a}_2\tilde{f}_1 - \tilde{e}_1\tilde{c}_2 + \tilde{b}_1\tilde{d}_2 & \hat{b}_1\hat{e}_2 - \hat{e}_1\hat{b}_2 & \hat{f}_1\hat{b}_2 - \hat{b}_1\hat{f}_2 & \hat{f}_1\hat{e}_2 - \hat{e}_1\hat{f}_2 \\ \hat{d}_1\hat{c}_2 - \hat{c}_1\hat{d}_2 & \tilde{a}_2\tilde{f}_1 - \tilde{e}_2\tilde{c}_1 + \tilde{b}_2\tilde{d}_1 & \hat{c}_1\hat{f}_2 - \hat{f}_1\hat{c}_2 & \hat{d}_1\hat{f}_2 - \hat{f}_1\hat{d}_2 \\ \hat{d}_1\hat{a}_2 - \hat{a}_1\hat{d}_2 & \hat{e}_1\hat{a}_2 - \hat{a}_1\hat{e}_2 & \tilde{a}_1\tilde{f}_2 - \tilde{e}_1\tilde{c}_2 + \tilde{b}_2\tilde{d}_1 & \hat{d}_1\hat{e}_2 - \hat{e}_1\hat{d}_2 \\ \hat{a}_1\hat{c}_2 - \hat{c}_1\hat{a}_2 & \hat{a}_1\hat{b}_2 - \hat{b}_1\hat{a}_2 & \hat{b}_1\hat{c}_2 - \hat{c}_1\hat{b}_2 & \tilde{a}_1\tilde{f}_2 - \tilde{e}_2\tilde{c}_1 + \tilde{b}_1\tilde{d}_2 \end{bmatrix},$$

where $\Delta_0 = \hat{a}_1\hat{f}_1 - \hat{c}_1\hat{e}_1 + \hat{d}_1\hat{b}_1$. ■

4.5 Examples

We will illustrate how to find the matrices for some types of Jacobi PDE's, with a few examples.

In all of the examples, we have given the matrixes in the basis $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right)$.

4.5.1 The symplectic Monge-Ampère equations

Let us take the symplectic Monge-Ampère equations which we classified in example (3.7.3):

$$\hat{a}+b\frac{\partial^2\varphi}{\partial x_1^2}-d\frac{\partial^2\varphi}{\partial x_2^2}-c\frac{\partial^2\varphi}{\partial x_2\partial x_1}+e\frac{\partial^2\varphi}{\partial x_1\partial x_2}+\check{a}\frac{\partial\varphi}{\partial x_1}+\tilde{a}\frac{\partial\varphi}{\partial x_2}+f\left(\frac{\partial^2\varphi}{\partial x_1^2}\frac{\partial^2\varphi}{\partial x_2^2}-\frac{\partial^2\varphi}{\partial x_2\partial x_1}\frac{\partial^2\varphi}{\partial x_1\partial x_2}\right)=0.$$

With the same substitutions as in example (3.7.3) , we obtain the system:

$$\begin{aligned} 0 &= \frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2}, \\ 0 &= a + b\frac{\partial h_1}{\partial x_1} - c\frac{\partial h_1}{\partial x_2} - d\frac{\partial h_2}{\partial x_2} + e\frac{\partial h_2}{\partial x_1} + f\left(\frac{\partial h_1}{\partial x_1}\frac{\partial h_2}{\partial x_2} - \frac{\partial h_1}{\partial x_2}\frac{\partial h_2}{\partial x_1}\right), \end{aligned}$$

where $a = \hat{a} + \check{a}h_1 + \tilde{a}h_2$.

Comparing this with the Jacobi PDE system $(*)$, we get that:

$$c_1 = 1, e_1 = 1, a_2 = a, b_2 = b, c_2 = c, d_2 = d, e_2 = e \text{ and } f_2 = f.$$

With this information we can find $\det(Q)$, which is equal $\mathcal{K}$:

$$\mathcal{K} = -4(af + bd) - (c - e)^2 .$$

Thus, we get the matrices of A_x, by inserting the information into the formula $(**)$:

$$||A_x|| = \left(\frac{\sqrt{|\,\mathcal{K}\,|}}{\mathcal{K}}\right) \begin{bmatrix} e-c & -2b & 0 & -2f \\ -2d & c-e & 2f & 0 \\ 0 & 2a & e-c & -2d \\ -2a & 0 & -2b & c-e \end{bmatrix}.$$

To verify the calculations, we calculate $||A_x^2||$, and get that:

$$||A_x^2|| = -\operatorname{sign}(\mathcal{K})\, 1,$$

which is what we could expect, since the symplectic Monge-Ampère equations are:

- elliptic, if $\mathcal{K} > 0$ then $||A_x^2|| = -1$,
- hyperbolic, if $\mathcal{K} < 0$ then $||A_x^2|| = 1$.

Laplace equation

Let us consider the Laplace equation:

$$\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} = 0,$$

and find its matrices $||A_x||$.

This is clearly a special case of the symplectic Monge-Ampère equations above. Thus, by inspection we get that:

$$b = 1,\ d = -1 \text{ and } \mathcal{K} = 4.$$

Hence:

$$||A_x|| = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Wave equation

Let us, as a second special case of the symplectic Monge-Ampère equations, analyze the Wave equation:

$$\frac{\partial^2 \varphi}{\partial x_1^2} - \frac{\partial^2 \varphi}{\partial x_2^2} = 0.$$

By inspection we get that:

$$b = 1, d = 1 \text{ and } \mathcal{K} = -4.$$

Thus:

$$||A_x|| = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Von Karman equation

Von Karman equation in the transonic approximation of gas dynamics, has the form:

$$\frac{\partial \varphi}{\partial x_1}\frac{\partial^2 \varphi}{\partial^2 x_1} - \frac{\partial^2 \varphi}{\partial^2 x_2} = 0,$$

where $\varphi = \varphi\,(x_1, x_2)$ is the velocity potential.

By inspection we get that:

$$b = \frac{\partial \varphi}{\partial x_1} = u_1, d = 1 \text{ and } \mathcal{K} = -4u_1,$$

and so we will have to assume that $u_1 \neq 0$, in order to have $\mathcal{K} \neq 0$.

The Von Karman equation has the following classification:

- elliptic, if $u_1 < 0$,
- hyperbolic, if $u_1 > 0$.

Thus, we get the matrix $||A_x||$:

$$||A_x|| = \left(\frac{\sqrt{|\, u_1 \,|}}{u_1}\right) \begin{bmatrix} 0 & u_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & u_1 & 0 \end{bmatrix},$$

and $||A_x^2|| = \text{sign}\,(u_1)\,1$.

4.5.2 A hyperbolic PDE system

Let us consider the system of Example (3.7.2) :

$$\begin{aligned} I \quad &: \quad h_2 \frac{\partial h_1}{\partial x_1} - \frac{\partial h_1}{\partial x_2} + \frac{1}{h_1 - h_2} = 0, \\ II \quad &: \quad h_1 \frac{\partial h_2}{\partial x_1} - \frac{\partial h_2}{\partial x_2} + \frac{1}{h_2 - h_1} = 0. \end{aligned}$$

As we remember, this system was hyperbolic at any point, since:

$$\mathcal{K} = -\left(u_2 - u_1\right)^2.$$

A little problem occurs when we calculate Q. It turns out that $\mathcal{A} = 0$, so we can not apply our results directly. We have assumed that $\mathcal{A} \neq 0$ when we constructed the orthogonal basis from which we make the operator A_x. The solution to this problem, is simply to consider the system $\{I + II, I\}$.

Hence we get:

$$b_1 = h_2\ , c_1 = 1, d_1 = 1, e_1 = u_1, a_2 = \frac{-1}{u_2\ - u_1}, b_2 = u_2\ , c_2 = 1,$$

and:

$$Q = \begin{bmatrix} 2\left(u_2\ - u_1\right) & u_2\ - u_1 \\ u_2\ - u_1 & 0 \end{bmatrix}.$$

Clearly $\mathcal{A} \neq 0$, and with comparison with $(**)$, we get:

$$||A_x|| = \frac{\sqrt{|\ \mathcal{K}\ |}}{\mathcal{K}} \begin{bmatrix} -u_2\ - u_1 & -2u_1u_2 & 0 & 0 \\ 2 & u_2\ + u_1 & 0 & 0 \\ \frac{-2}{u_2-u_1} & \frac{-2u_1}{u_2-u_1} & u_2\ - u_1 & 0 \\ \frac{2}{u_2-u_1} & \frac{2u_2}{u_2-u_1} & 0 & u_1 - u_2 \end{bmatrix},$$

and $||A_x^2|| = 1$.

Chapter 5

From operators to Jacobi planes

To complete the triangle from the introduction, we show that if we start off with a complex structure or a symmetric product structure, we are able to find its corresponding elliptic or hyperbolic two-dimensional plane Π in $\Lambda^2(V^*)$.

For simplicity, we make the following changes in our notation $\theta_2 = \omega$, $\theta_1 = \Omega$.

We introduce the operator $\Lambda^2\left(A^*\right)$ on $\Lambda^2\left(V^*\right)$:

$$\begin{aligned} \Lambda^2\left(A^*\right) : \Lambda^2\left(V^*\right) &\longrightarrow \Lambda^2\left(V^*\right), \\ \theta(*,*) &\longmapsto \theta(A*,A*). \end{aligned}$$

Proposition 5.1 *The two 2-forms Ω and ω, are eigenvectors of the operator $\Lambda^2\left(A^*\right)$, that is:*

$$\begin{aligned} \Lambda^2\left(A^*\right)(\Omega) &= \varepsilon(\Pi)\,\Omega, \\ \Lambda^2\left(A^*\right)(\omega) &= \varepsilon(\Pi)\,\omega. \end{aligned}$$

Proof. Since:

$$\omega(X,Y) = \Omega(AX,Y),$$

and we deduce that:

$$\begin{aligned}
\Lambda^2(A^*)(\Omega)(AX,Y) &= \Omega(A^2X,AY) = \Omega(\varepsilon(\Pi)X,AY) \\
&= \varepsilon(\Pi)\,\Omega(X,AY) = -\varepsilon(\Pi)\,\Omega(AY,X) \\
&= -\varepsilon(\Pi)\,\omega(Y,X) = \varepsilon(\Pi)\,\omega(X,Y) = \varepsilon(\Pi)\,\Omega(AX,Y).
\end{aligned}$$

Hence:

$$\Lambda^2(A^*)(\Omega) = \varepsilon(\Pi)\,\Omega.$$

In a similar way, we get:

$$\begin{aligned}
\Lambda^2(A^*)(\omega)(AX,Y) &= \omega(A^2X,AY) = \varepsilon(\Pi)\,\omega(X,AY) \\
&= \varepsilon(\Pi)\,\Omega(AX,AY) = -\varepsilon(\Pi)\,\Omega(AY,AX) \\
&= -\varepsilon(\Pi)\,\Omega(AY,AX) = -\varepsilon(\Pi)\,\omega(Y,AX) = \varepsilon(\Pi)\,\omega(AX,Y).
\end{aligned}$$

Therefore:

$$\Lambda^2(A^*)(\omega) = \varepsilon(\Pi)\,\omega.$$

■

5.1 Symmetric product structures

Let (V,A) be a symmetric product structure, and V a four-dimensional symplectic vector space.

Note that:

$$A^2 = 1 \Leftrightarrow \frac{1}{2}(1-A)\frac{1}{2}(1+A) = 0.$$

Moreover $\frac{1}{2}(1+A)$ and $\frac{1}{2}(1-A)$ are **projectors** in V, and denote the projector $\frac{1}{2}(1+A)$

by P_+, and the projector $\frac{1}{2}(1-A)$ by P_-, and $\operatorname{im} P_+ = V_+, \operatorname{im} P_- = V_-$, that is:

$$\begin{matrix} P_+ : V \longrightarrow V_+, \\ P_+^2 = P_+, \end{matrix} \quad \text{and} \quad \begin{matrix} P_- : V \longrightarrow V_-, \\ P_-^2 = P_- \end{matrix} .$$

Let X be a vector from V, since$1 = P_+ + P_-$, then $X = P_+X + P_-X$, and we will denote P_+X by X_+, and P_-X by X_-.

Clearly:

$$P_+(Y) = P_+(AY) \text{ and } P_-(X) = P_-(-AX),$$

for $Y \notin V_-$ and $X \notin V_+$.

Proposition 5.2 *The product structure* (V, A), *produces a splitting of the four dimensional vector space* V *into* $V_+ \oplus V_-$.

Proof. The projectors P_+ and P_- produce a splitting of V into $V_+ \oplus V_-$, since:

i) $\operatorname{im} P_+ = V_+$ and $\operatorname{im} P_- = V_-$,

ii) $1 = P_+ + P_-$,

iii) $P_+ \circ P_- = P_- \circ P_+ = 0$.

■

Proposition 5.3 *The vector space* V_+ *is skew-orthogonal on* V_- *with respect to* Ω *and* ω. *That is:*

$$\Omega(X_-, Y_+) = 0 \text{ and } \omega(X_-, Y_+) = 0.$$

Proof. First we see that:

$$\omega(X, Y) = \Omega(AX, Y) \Longleftrightarrow \omega(AX, Y) = \Omega(X, Y).$$

Since Y and AY go to the same element in V_+ , and X and $-AX$ go to the same element in V_-, we derive that:

$$\Omega(X_-, Y_+) = \Omega(-AX_-, AY_+) = -\omega(A^2X_-, AY_+) = \omega(AY_+, X_-) = -\Omega(X_-, Y_+).$$

Hence:

$$\Omega(X_-, Y_+) = 0.$$

And similarly for $\omega(X_-, Y_+)$:

$$\omega(X_-, Y_+) = \omega(-AX_-, AY_+) = -\omega(AX_-, AY_+) = -\Omega(X_-, AY_+) = -\omega(X_-, Y_+),$$

so:

$$\omega(X_-, Y_+) = 0.$$

∎

Proposition 5.4 *Any hyperbolic Jacobi plane* Π, *determines a symmetric product structure* (V, A).

Proof. We have already shown that it determines a product structure (V, A), so we shall only show that:

$$\dim(V_+) = \dim(V_-).$$

The only two cases we will have to investigate are when, $\dim(V_+) = 1$,$\dim(V_-) = 3$, and $\dim(V_+) = 0$,$\dim(V_-) = 4$.

Assume that $\dim(V_+) = 1$ and $\dim(V_-) = 3$.

Then, due to proposition (5.3), we know that $\Omega(V_+, V_-) = 0$ and $\Omega(V_+, V_+) = 0$, since V_+ is one-dimensional. This implies that Ω is a degenerated form, which is a contradiction.

Assume that $\dim(V_+) = 0$ and $\dim(V_-) = 4$, then A has to be equal -1.

Since $\omega(X,Y) = \Omega(AX,Y)$, we get that $\omega = -\Omega$, which is a contradiction, since ω and Ω do not generate a plane.

So:

$$\dim(V_+) = \dim(V_-) = 2.$$

■

Note that $\Omega|_{V_-}$ and $\Omega|_{V_+}$ are non-degenerated, and $\omega|_{V_-}$ and $\omega|_{V_+}$ are non-degenerated.

We will show that there exists a basis on $V = \langle e_{1,}e_2, f_1, f_2\rangle$, such that the forms $\Omega, \omega \in \Lambda^2(V^*)$ can be written as:

$$\Omega = e_1^* \wedge f_1^* + e_2^* \wedge f_2^* \text{ and } \omega = e_1^* \wedge f_1^* - e_2^* \wedge f_2^*.$$

Choose $e_1 \in V_+$. Since $\Omega(X,Y)|_{V_+}$ is non-degenerated, there exists an $\tilde{f}_1 \in V_+$, such that $\Omega(e_1, \tilde{f}_1) \neq 0$.

Then we choose $f_1 = \frac{\tilde{f}_1}{\Omega(e_1,\tilde{f}_1)}$, and get that:

$$\Omega(e_1, f_1) = 1.$$

We apply the similar procedure for V_-.

Choose $e_2 \in V_-$. Since $\Omega(X,Y)|_{V_-}$ is non-degenerated, there exists an $\tilde{f}_2 \in V_-$, such that $\Omega(e_2, \tilde{f}_2) \neq 0$. Then we choose $f_2 = \frac{\tilde{f}_2}{\Omega(e_2,\tilde{f}_2)}$, and get that:

$$\Omega(e_2, f_2) = 1.$$

From this we derive the following:

$$\begin{aligned}
\Omega(e_1, f_1) &= 1 \\
&= \Omega(Ae_1, f_1) \quad \text{since } e_1 \in V_+ \\
&= \omega(e_1, f_1) \\
\omega(e_1, f_1) &= 1.
\end{aligned}$$

And similarly:

$$\begin{aligned}
\Omega(e_2, f_2) &= 1 \\
&= \Omega(-Ae_2, f_2) \ \text{ since } e_2 \in V_- \\
&= -\omega(e_2, f_2) \\
\omega(e_2, f_2) &= -1.
\end{aligned}$$

The result of this procedure is:

$$\begin{aligned}
\Omega &= e_1^* \wedge f_1^* + e_2^* \wedge f_2^*, \\
\omega &= e_1^* \wedge f_1^* - e_2^* \wedge f_2^*,
\end{aligned}$$

and:

$$\begin{array}{rccc}
A: & V & \longrightarrow & V \\
 & e_1 & \longmapsto & e_1 \\
 & e_2 & \longmapsto & -e_2 \\
 & f_1 & \longmapsto & f_1 \\
 & f_2 & \longmapsto & -f_2
\end{array}.$$

Therefore, if we know the decomposition of $V = V_+ \oplus V_-$, then we can construct Ω, ω and A on the form above.

Let $B : V \longrightarrow V$ be an operator. We introduce the operator i_B as an inner derivation on $\Lambda^2(V^*)$, by:

$$\begin{array}{rccc}
i_B: & \Lambda^2(V^*) & \longrightarrow & \Lambda^2(V^*), \\
 & \theta(*,*) & \longmapsto & \theta(B*,*) + \theta(*,B*).
\end{array}$$

Proposition 5.5 *Let (V, A) be the symmetric product structure derived from a hyperbolic Jacobi plane $\Pi = \langle \omega, \Omega \rangle$, then:*

$$i_A\omega = 2\Omega,$$

and:

$$i_A\Omega = 2\omega.$$

Proof. When we compute $i_A\omega(X,Y)$, we get that:

$$i_A\omega(X,Y) = \omega(AX,Y) + \omega(X,AY) = \Omega(A^2X,Y) - \Omega(A^2Y,X) = 2\Omega(X,Y).$$

And for $i_A\Omega(X,Y)$, we get that:

$$i_A\Omega(X,Y) = \Omega(AX,Y) + \Omega(X,AY) = \omega(X,Y) - \omega(Y,X) = 2\omega(X,Y).$$

■

Theorem 5.6 *The image of i_A on the space $\Lambda^2(V^*)$ is the plane $\Pi = \langle\omega,\Omega\rangle \subset \Lambda^2(V^*)$. The spectrum of i_A is $\{-2,0,0,0,0,2\}$. Moreover, $\Omega+\omega$ is an eigenvector for the eigenvalue 2, and $\Omega-\omega$ is an eigenvector for the eigenvalue -2.*

Proof. Take $\{e_1^*\wedge f_1^*, e_2^*\wedge f_1^*, e_2^*\wedge e_1^*, f_2^*\wedge e_1^*, f_2^*\wedge f_1^*, e_2^*\wedge f_2^*\}$ as a basis for $\Lambda^2(V^*)$:

$$\begin{array}{rccl} i_A: & \Lambda^2(V^*) & \longrightarrow & \Lambda^2(V^*) \\ & e_1^*\wedge f_1^* & \longmapsto & 2e_1^*\wedge f_1^* \\ & e_2^*\wedge f_1^* & \longmapsto & 0 \\ & e_2^*\wedge e_1^* & \longmapsto & 0 \\ & f_2^*\wedge e_1^* & \longmapsto & 0 \\ & f_2^*\wedge f_1^* & \longmapsto & 0 \\ & e_2^*\wedge f_2^* & \longmapsto & -2e_2^*\wedge f_2^* \end{array}.$$

Let Ω, ω be the normal forms given by:

$$\Omega = e_1^*\wedge f_1^* + e_2^*\wedge f_2^* \text{ and } \omega = e_1^*\wedge f_1^* - e_2^*\wedge f_2^*.$$

Clearly $\{\Omega+\omega, \Omega-\omega, e_2^*\wedge f_1^*, e_2^*\wedge e_1^*, f_2^*\wedge e_1^*, f_2^*\wedge f_1^*\}$ is also a basis for $\Lambda^2(V^*)$.

Due to the proposition (5.5), we know that $i_A(\Omega) = 2\omega$ and $i_A(\omega) = 2\Omega$. So we get:

$$\begin{aligned} i_A(\Omega+\omega) &= 2(\Omega+\omega), \\ i_A(\Omega-\omega) &= -2(\Omega-\omega). \end{aligned}$$

■

5.2 Complex structures

Let V be a real vector space. The tensor product $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector space called the **complexification** of V.

Every element in the complex vector space $V^{\mathbb{C}}$ can be written uniquely as a sum $v + iv'$ with $v, v' \in V$.

Using the real direct sum decomposition, we get the following canonical isomorphism:

$$V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV,$$

since $V \otimes_{\mathbb{R}} \mathbb{C} = \langle v \otimes_{\mathbb{R}} 1, v \otimes_{\mathbb{R}} i \rangle$.

Let V be a real vector space with a linear operator $A : V \longrightarrow V$, then $A^{\mathbb{C}}$ (the complexification of A) is a $\mathbb{C}$-linear operator on $V^{\mathbb{C}}$, and it acts like:

$$\begin{aligned} A^{\mathbb{C}} &: V^{\mathbb{C}} \longrightarrow V^{\mathbb{C}}, \\ A^{\mathbb{C}}(v \otimes_{\mathbb{R}} \lambda) &= A(v) \otimes_{\mathbb{R}} \lambda, \end{aligned}$$

where $\lambda \in \mathbb{C}$.

Let (A, V) be a complex structure and $A^2 = -1$.

When we make the complexification A, V and of $A^2 = -1$, we get:

$$\left(A^{\mathbb{C}}, V^{\mathbb{C}}\right) \text{ and } \left(A^{\mathbb{C}}\right)^2 = -1.$$

Since V is a four dimensional real vector space, then $V^{\mathbb{C}}$ is a four-dimensional complex vector space.

Note that:

$$\left(A^{\mathbb{C}}\right)^2 = -1 \Longleftrightarrow \frac{1}{2}(1 - iA^{\mathbb{C}})\frac{1}{2}(1 + iA^{\mathbb{C}}) = 0.$$

Moreover, $\frac{1}{2}(1+iA^{\mathbb{C}})$ and $\frac{1}{2}(1-iA^{\mathbb{C}})$ are **projectors** in $V^{\mathbb{C}}$, and denote the projector $\frac{1}{2}(1+iA^{\mathbb{C}})$ by P_+, and the projector $\frac{1}{2}(1-iA^{\mathbb{C}})$ by P_- and $\operatorname{im} P_+ = V_+^{\mathbb{C}}, \operatorname{im} P_- = V_-^{\mathbb{C}}$, that is:

$$\begin{matrix} P_+ : V^{\mathbb{C}} \longrightarrow V_+^{\mathbb{C}}, \\ (P_+)^2 = P_+, \end{matrix} \quad \text{and} \quad \begin{matrix} P_- : V^{\mathbb{C}} \longrightarrow V_-^{\mathbb{C}}, \\ (P_-)^2 = P_-. \end{matrix} \, .$$

Let X be a vector from $V^{\mathbb{C}}$, and since:

$$1 = P_+ + P_-,$$

then $X = P_+X + P_-X$, and we will denote P_+X by X_+ and P_-X by X_-.

Proposition 5.7 *The complex structure* (V, A) , *produces a splitting of complexification of* $V^{\mathbb{C}}$:

$$V^{\mathbb{C}} = V_+^{\mathbb{C}} \oplus V_-^{\mathbb{C}}.$$

Proof. We will prove this in the same way as we did for the hyperbolic case.

The projectors P_+ and P_- produces a splitting of $V^{\mathbb{C}}$ into $V_+^{\mathbb{C}} \oplus V_-^{\mathbb{C}}$, since:

i) $\operatorname{im} P_+ = V_+^{\mathbb{C}}$ and $\operatorname{im} P_- = V_-^{\mathbb{C}}$,

ii) $1 = P_+ + P_-$,

iii) $P_+ \circ P_- = P_- \circ P_+ = 0$.

■

We also note that:

$$\begin{aligned} A^{\mathbb{C}} X_{+} &= -iX_{+}, \\ A^{\mathbb{C}} X_{-} &= iX_{-}, \end{aligned}$$

since:

$$\begin{aligned} \left(A^{\mathbb{C}}\right) \frac{1}{2}(1 + iA^{\mathbb{C}}) &= \frac{1}{2}(-i + A^{\mathbb{C}}) = -i\frac{1}{2}(1 + iA^{\mathbb{C}}), \\ \left(A^{\mathbb{C}}\right) \frac{1}{2}(1 - iA^{\mathbb{C}}) &= \frac{1}{2}(i + A^{\mathbb{C}}) = i\frac{1}{2}(1 - iA^{\mathbb{C}}). \end{aligned}$$

Proposition 5.8 *For any non-zero vectors $Y \notin V_{+}^{\mathbb{C}}$ and $X \notin V_{-}^{\mathbb{C}}$, we have that:*

$$\begin{aligned} V_{-}^{\mathbb{C}} &= \langle P_{-}(Y), P_{-}\left(A^{\mathbb{C}} Y\right)\rangle, \\ V_{+}^{\mathbb{C}} &= \langle P_{+}(X), P_{+}\left(A^{\mathbb{C}} X\right)\rangle. \end{aligned}$$

Proof. All we need to show, is that $P_{+}(X) \neq P_{+}\left(A^{\mathbb{C}} X\right)$ and $P_{-}(Y) \neq P_{-}\left(A^{\mathbb{C}} Y\right)$.
Proof by contradiction:
Assume that:

$$\begin{aligned} P_{-}\left(A^{\mathbb{C}} Y\right) &= P_{-}(Y) \\ iP_{-}(Y) &= P_{-}(Y). \end{aligned}$$

Let $P_{-}(Y)$ be equal to $v \otimes_{\mathbb{R}} (x + iy)$, hence:

$$\begin{aligned} i\left(v \otimes_{\mathbb{R}} x + iy\right) &= \left(v \otimes_{\mathbb{R}} x + iy\right) \\ v \otimes_{\mathbb{R}} (-y + xi) &= v \otimes_{\mathbb{R}} (x + iy). \end{aligned}$$

Therefore $x = -y$ and $x = y$, which is a contradiction unless $x = y = 0$.
In a similar way we can show that $P_{+}(X) \neq P_{+}\left(A^{\mathbb{C}} X\right)$. ■

Let $\theta \in \Lambda^2(V^*)$ be a real valued 2-form. We define the complexification of θ by $\theta^{\mathbb{C}} \in \Lambda^2(V^{\mathbb{C}*})$:

$$\begin{aligned}\theta^{\mathbb{C}} : V^{\mathbb{C}} \times V^{\mathbb{C}} &\longrightarrow \mathbb{C}, \\ \theta^{\mathbb{C}}\left(v \otimes_{\mathbb{R}} \alpha, u \otimes_{\mathbb{R}} \beta\right) &= \theta\left(v, u\right) \alpha\beta.\end{aligned}$$

In a similar way as we did for the symmetric product structure, we derive the following:

Proposition 5.9 *The vector space $V_+^{\mathbb{C}}$ is skew-orthogonal on $V_-^{\mathbb{C}}$, with respect to $\Omega^{\mathbb{C}}$ and $\omega^{\mathbb{C}}$. That is:*

$$\Omega^{\mathbb{C}}(X_-, Y_+) = 0 \text{ and } \omega^{\mathbb{C}}(X_-, Y_+) = 0.$$

Note that $\Omega^{\mathbb{C}}|_{V_-^{\mathbb{C}}}$ and $\Omega^{\mathbb{C}}|_{V_+^{\mathbb{C}}}$ are non-degenerated, and $\omega^{\mathbb{C}}|_{V_-^{\mathbb{C}}}$ and $\omega^{\mathbb{C}}|_{V_+^{\mathbb{C}}}$ are non-degenerated.

Theorem 5.10 *There exists a basis on $V^{\mathbb{C}} = \langle e_1, e_2, f_1, f_2 \rangle$, such that the forms $\Omega^{\mathbb{C}}, \omega^{\mathbb{C}} \in \Lambda^2\left(V^{\mathbb{C}*}\right)$ can be written as:*

$$\Omega^{\mathbb{C}} = e_1^* \wedge f_1^* + e_2^* \wedge f_2^* \text{ and } \omega^{\mathbb{C}} = ie_2^* \wedge f_2^* - ie_1^* \wedge f_1^*,$$

and we will call them the ***normal forms*** *for the plane $\Pi^{\mathbb{C}} = \left\langle \Omega^{\mathbb{C}}, \omega^{\mathbb{C}} \right\rangle$.*

Proof. Choose $e_1 \in V_+^{\mathbb{C}}$. Since $\Omega^{\mathbb{C}}|V_+^{\mathbb{C}}$ is non-degenerated, there exists an $\tilde{f}_1 \in V_+^{\mathbb{C}}$, such that $\Omega^{\mathbb{C}}(e_1, \tilde{f}_1) \neq 0$.

Therefore we choose $f_1 = \frac{\tilde{f}_1}{\Omega(e_1, \tilde{f}_1)}$, and $\Omega^{\mathbb{C}}(e_1, f_1) = 1$

Choose $e_2 \in V_-^{\mathbb{C}}$. Since $\Omega^{\mathbb{C}}|V_-^{\mathbb{C}}$ is non-degenerated, there exists an $\tilde{f}_2 \in V_-^{\mathbb{C}}$, such that $\Omega^{\mathbb{C}}(e_2, \tilde{f}_2) \neq 0$.

Therefore we choose $f_2 = \frac{\tilde{f}_2}{\Omega(e_2, \tilde{f}_2)}$, and $\Omega^{\mathbb{C}}(e_2, f_2) = 1$.

As we have shown $-iX_+ = A^{\mathbb{C}}X_+$, $iX_- = A^{\mathbb{C}}X_-$ and:

$$\Omega^{\mathbb{C}}(X, Y) = -\omega^{\mathbb{C}}(A^{\mathbb{C}}X, Y).$$

Hence:

$$\begin{aligned}
\Omega^{\mathbb{C}}(e_1, f_1) &= 1 \\
&= -\omega^{\mathbb{C}}(Ae_1, f_1) \text{ since } e_1 \in V_+ \\
&= i\omega^{\mathbb{C}}(e_1, f_1).
\end{aligned}$$

Which means that:

$$\omega^{\mathbb{C}}(e_1, f_1) = -i.$$

In a similar way we get:

$$\omega^{\mathbb{C}}(e_2, f_2) = i.$$

The result of this is:

$$\begin{aligned}
\Omega^{\mathbb{C}} &= e_1^* \wedge f_1^* + e_2^* \wedge f_2^*, \\
\omega^{\mathbb{C}} &= ie_2^* \wedge f_2^* - ie_1^* \wedge f_1^*,
\end{aligned}$$

and:

$$\begin{array}{rccc}
A^{\mathbb{C}}: & V^{\mathbb{C}} & \longrightarrow & V^{\mathbb{C}} \\
& e_1 & \longmapsto & -ie_1 \\
& e_2 & \longmapsto & ie_2 \\
& f_1 & \longmapsto & -if_1 \\
& f_2 & \longmapsto & if_2
\end{array}.$$

■

Therefore, if we know the decomposition of $V^{\mathbb{C}} = V_+^{\mathbb{C}} \oplus V_-^{\mathbb{C}}$, then we can construct $\Omega^{\mathbb{C}}, \omega^{\mathbb{C}}$ and $A^{\mathbb{C}}$ on the form above.

Let $B : V \longrightarrow V$ be an $\mathbb{R}$-linear operator, and $\theta \in \Lambda^2(V^*)$ a 2-form. We introduce the operator $i_{B^{\mathbb{C}}}$ as an inner derivation on $\Lambda^2\left(V^{\mathbb{C}*}\right)$, by:

$$\begin{array}{rccl} i_{B^{\mathbb{C}}}: & \Lambda^2\left(V^{\mathbb{C}*}\right) & \longrightarrow & \Lambda^2\left(V^{\mathbb{C}*}\right), \\ & \theta^{\mathbb{C}}(*,*) & \longmapsto & \theta^{\mathbb{C}}(B^{\mathbb{C}}*,*)+\theta^{\mathbb{C}}(*,B^{\mathbb{C}}*). \end{array}$$

where $B^{\mathbb{C}}$ is the complexification of B, and $\theta^{\mathbb{C}}$ is the complexification of θ.

Proposition 5.11 *Let (V, A) be the complex structure derived from an elliptic Jacobi plane $\Pi = \langle \omega, \Omega \rangle$, then:*

$$i_{A^{\mathbb{C}}}\omega^{\mathbb{C}} = -2\Omega^{\mathbb{C}},$$

and:

$$i_{A^{\mathbb{C}}}\Omega^{\mathbb{C}} = 2\omega^{\mathbb{C}}.$$

Proof. When we compute $i_{A^{\mathbb{C}}}\omega^{\mathbb{C}}(X,Y)$, we get that:

$$\begin{aligned} i_{A^{\mathbb{C}}}\omega^{\mathbb{C}}(X,Y) &= \omega^{\mathbb{C}}(A^{\mathbb{C}}X,Y)+\omega^{\mathbb{C}}(X,A^{\mathbb{C}}Y)=\Omega^{\mathbb{C}}(\left(A^{\mathbb{C}}\right)^2X,Y)-\Omega^{\mathbb{C}}(\left(A^{\mathbb{C}}\right)^2Y,X) \\ &= -2\Omega^{\mathbb{C}}(X,Y). \end{aligned}$$

And for $i_{A^{\mathbb{C}}}\Omega^{\mathbb{C}}(X,Y)$, we get that:

$$i_{A^{\mathbb{C}}}\Omega^{\mathbb{C}}(X,Y)=\Omega^{\mathbb{C}}(A^{\mathbb{C}}X,Y)+\Omega(X,A^{\mathbb{C}}Y)=\omega^{\mathbb{C}}(X,Y)-\omega^{\mathbb{C}}(Y,X)=2\omega^{\mathbb{C}}(X,Y).$$

■

Theorem 5.12 *The image of the operator $i_{A^{\mathbb{C}}}$ on the space $\Lambda^2(V^{\mathbb{C}*})$, is the plane $\Pi^{\mathbb{C}} = \left\langle \omega^{\mathbb{C}}, \Omega^{\mathbb{C}} \right\rangle$. The spectrum of $i_{A^{\mathbb{C}}}$ is $\{-2i, 0, 0, 0, 0, 2i\}$, and $(\Omega^{\mathbb{C}} + i\omega^{\mathbb{C}})$ is the eigenvector corresponding to the eigenvalue $-2i$, and $(\Omega^{\mathbb{C}} - i\omega^{\mathbb{C}})$ is the eigenvector corresponding to the eigenvalue $2i$.*

Proof. Take $\{e_1^* \wedge f_1^* + e_2^* \wedge f_2^*, e_2^* \wedge f_2^* - e_1^* \wedge f_1^*, e_2^* \wedge f_1^*, e_2^* \wedge e_1^*, f_2^* \wedge e_1^*, f_2^* \wedge f_1^*\}$ as a basis for $\Lambda^2(V^{\mathbb{C}*})$:

$$\begin{array}{rlcl} i_{A^{\mathbb{C}}}: & \Lambda^2(V^{\mathbb{C}*}) & \longrightarrow & \Lambda^2(V^{\mathbb{C}*}) \\ & e_1^* \wedge f_1^* + e_2^* \wedge f_2^* & \longmapsto & 2ie_2^* \wedge f_2^* - 2ie_1^* \wedge f_1^* = 2\omega^{\mathbb{C}} \\ & e_2^* \wedge f_1^* & \longmapsto & 0 \\ & e_2^* \wedge e_1^* & \longmapsto & 0 \\ & f_2^* \wedge e_1^* & \longmapsto & 0 \\ & f_2^* \wedge f_1^* & \longmapsto & 0 \\ & e_2^* \wedge f_2^* - e_1^* \wedge f_1^* & \longmapsto & 2ie_1^* \wedge f_1^* + 2ie_2^* \wedge f_2^* = 2i\Omega^{\mathbb{C}} \end{array} .$$

Therefore:

$$\begin{aligned} i_{A^{\mathbb{C}}}(\Omega^{\mathbb{C}} + i\omega^{\mathbb{C}}) &= 2\omega^{\mathbb{C}} - i2\Omega^{\mathbb{C}} = -2i(\Omega^{\mathbb{C}} + i\omega^{\mathbb{C}}), \\ i_{A^{\mathbb{C}}}(\Omega^{\mathbb{C}} - i\omega^{\mathbb{C}}) &= 2\omega^{\mathbb{C}} + i2\Omega^{\mathbb{C}} = 2i(\Omega^{\mathbb{C}} - i\omega^{\mathbb{C}}). \end{aligned}$$

■

Lemma 5.13 *Let $B : V \longrightarrow W$ be a $\mathbb{R}$-linear operator from the vector space V to the vector space W, and let $B^{\mathbb{C}} : V^{\mathbb{C}} \longrightarrow W^{\mathbb{C}}$ be the complexification. Then, the image of B complexified is equal to the image of $B^{\mathbb{C}}$, that is:*

$$(\operatorname{Im} B)^{\mathbb{C}} = \operatorname{Im} B^{\mathbb{C}}.$$

Proof. Let us show that $(\operatorname{Im} B)^{\mathbb{C}} \supset \operatorname{Im} B^{\mathbb{C}}$.
Assume that: $x + iy \in (\operatorname{Im} B)^{\mathbb{C}}$, where $x, y \in \operatorname{Im} B$, and that $x = Bv_0, y = Bv_1$.
Hence:

$$\begin{aligned} x + iy &= Bv_0 + iBv_1 \\ &= B^{\mathbb{C}}(v_0 + iv_1) \in \operatorname{Im} B^{\mathbb{C}}. \end{aligned}$$

Let us then show that $\operatorname{Im} B^{\mathbb{C}} \supset (\operatorname{Im} B)^{\mathbb{C}}$.

Let $x + iy \in \operatorname{Im} B^{\mathbb{C}}$, then there exists $v_0 + iv_1$, where $v_0, v_1 \in V$, such that:

$$\begin{aligned} x + iy &= B^{\mathbb{C}} (v_0 + iv_1) \\ &= Bv_0 + iBv_1, \end{aligned}$$

hence $x = Bv_{0,}$ and $y = Bv_1$, and so $x + iy \in (\operatorname{Im} B)^{\mathbb{C}}$. ■

Due to the lemma above, we deduce that:

$$\begin{aligned} Im(i_{A^{\mathbb{C}}}) &= (Im\,(i_A))^{\mathbb{C}}, \\ \Pi^{\mathbb{C}} &= (\langle \omega, \Omega \rangle = \Pi)^{\mathbb{C}}. \end{aligned}$$

Therefore, we sum up the results in this section by:

Theorem 5.14 *Let (A, V) be a complex structure, then the image of i_A on the space $\Lambda^2(V^*)$, is the elliptic plane $\Pi = \langle \omega, \Omega \rangle$.*

So we found an operator, namely $i_A : \Lambda^2(V^*) \longrightarrow \Lambda^2(V^*)$, such that, if we start with a complex structure or a symmetric product structure, we are able to find the corresponding elliptic or hyperbolic two-dimensional plane by $\operatorname{im}(i_A) = \Pi$.

Chapter 6

Local classification of the Jacobi PDE system

The classification problem

We say that two Jacobi PDE systems Π and Π' on M are locally equivalent at the point $x \in M$, if there exist a local diffeomorphism $\varphi : U_x \longrightarrow V_x, \varphi(x) = x$, such that:

$$\varphi^*(\omega') \in \Pi,$$

for all $\omega' \in \Pi'$.

This means that the two Jacobi PDE systems have an isomorphic (by φ) space of solutions. That is, if L is the solution of Π, then $\varphi(L)$ is a solution of Π', and vice versa.

6.1 Local classification of hyperbolic Jacobi PDE's

In this section we will investigate when a hyperbolic Jacobi PDE system is locally equivalent to the wave system.

Let Π be a hyperbolic Jacobi PDE system. Then this system determines a symmetric almost product structure $A_x : T_xM \longrightarrow T_xM$ on M.

The symmetric almost product structure, determines a splitting of $T_xM = \mathcal{C}_{+,x} \oplus \mathcal{C}_{-,x}$, where $\mathcal{C}_+$and $\mathcal{C}_-$ are 2-dimensional distributions on M.

Consider for example the Wave system:

$$\begin{aligned} \frac{\partial h_1}{\partial x_2} - \frac{\partial h_2}{\partial x_1} &= 0, \\ \frac{\partial h_1}{\partial x_2} + \frac{\partial h_2}{\partial x_1} &= 0, \end{aligned}$$

and obtain the matrix representation:

$$||A|| = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

in the basis $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right)$.

For convenience, we make the following coordinate change:

$$(x_1, x_2, u_1, u_2,) \longmapsto (u_2 := q_1, x_2 := q_2, u_1 := p_1, x_1 := p_2).$$

Hence:

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right) \longmapsto \left(\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}\right),$$

and so the matrix will look like:

$$||A|| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{6.1}$$

in the basis$\left(\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}\right)$.

Then $\mathcal{C}_+$ is generated by $\frac{\partial}{\partial q_1}$ and $\frac{\partial}{\partial q_2}$, and $\mathcal{C}_-$ is generated by $\frac{\partial}{\partial p_1}$ and $\frac{\partial}{\partial p_2}$.

We see that both $\mathcal{C}_+$ and $\mathcal{C}_-$ are completely integrable distributions.

Theorem 6.1 *Let Π be a hyperbolic Jacobi PDE system, such that $\mathcal{C}_+$ and $\mathcal{C}_-$ are completely integrable distributions. Then Π is locally equivalent to the Wave system.*

Proof. Let p_1 and p_2 be first integrals for $\mathcal{C}_+$, and q_1 and q_2 be first integrals for $\mathcal{C}_-$.

Note that $\frac{\partial}{\partial q_1}$ and $\frac{\partial}{\partial q_1}$ are tangent to $\mathcal{C}_+$, because $\frac{\partial}{\partial q_1}(p_1) = \frac{\partial}{\partial q_2}(p_2) = 0$, and that $\frac{\partial}{\partial p_1}$ and $\frac{\partial}{\partial p_2}$ are tangent to $\mathcal{C}_-$, because $\frac{\partial}{\partial p_1}(q_1) = \frac{\partial}{\partial p_2}(q_2) = 0$.

In these coordinates one has:

$$\begin{aligned} A &: \frac{\partial}{\partial q_i} \longmapsto \frac{\partial}{\partial q_i}, \\ A &: \frac{\partial}{\partial q_i} \longmapsto -\frac{\partial}{\partial p_i}. \end{aligned}$$

And so it coincides with the Wave system above. ■

We now want to make use of the operator representation for the hyperbolic Jacobi PDE systems, to find a criterion for when $\mathcal{C}_+$ and $\mathcal{C}_-$ are completely integrable.

To do this we will need the Nijenhuis tensor.

The Nijenhuis tensor $|[B,B]|$ is equal to:

$$|[B,B]|(X,Y) = [BX,BY] - B[BX,Y] - B[X,BY] + B^2[X,Y],$$

where $B \in \Omega^1(M) \otimes \mathcal{D}(M)$ and $X, Y \in \mathcal{D}(M)$.

Theorem 6.2 (Main theorem 1) *Let Π be a hyperbolic Jacobi PDE system. Then Π is locally equivalent to the Wave system if and only if:*

$$|[A,A]| = 0.$$

Proof. If Π is the Wave equation on the form described above, then due to (6.1), we have that $|[A,A]| = 0$.

On the other hand if $|[A,A]| = 0$, will prove that the distributions $\mathcal{C}_+$ and $\mathcal{C}_-$ are completely integrable.

Assume that X, Y are smooth vector fields from $\mathcal{C}_+$, then:

$$\begin{aligned} 0 &= |[A,A]|\,(X,Y) \\ &= [AX,AY] - A\,[AX,Y] - A\,[X,AY] + [X,Y]\,. \end{aligned}$$

Since $AY = Y$ in $\mathcal{C}_+$, we get that:

$$0 = [X,Y] - A\,[X,Y] - A\,[X,Y] + [X,Y]\,,$$

or

$$(1-A)\,[X,Y] = 0.$$

Therefore, $[X,Y]$ belongs to $\mathcal{C}_+$, and $\mathcal{C}_+$ is completely integrable.

In a similar way we may prove that $\mathcal{C}_-$ is completely integrable. ■

6.2 Local classification of elliptic Jacobi PDE's

The classification problem we will solve in this section is when an elliptic Jacobi PDE system is locally equivalent to the Cauchy-Riemann system.

Let us first consider the Cauchy-Riemann system:

$$\begin{aligned} \frac{\partial h_2}{\partial x_1} - \frac{\partial h_1}{\partial x_2} &= 0, \\ \frac{\partial h_1}{\partial x_1} + \frac{\partial h_2}{\partial x_2} &= 0. \end{aligned}$$

As we have seen, the matrix of the operator A is:

$$||A|| = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{6.2}$$

in the basis $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right)$.

Theorem 6.3 (Main theorem 2) *Let* Π *be an elliptic Jacobi PDE system. Then* Π *is locally equivalent to the Cauchy-Riemann system if and only if:*

$$|[A, A]| = 0.$$

Proof. If the Cauchy-Riemann system is on the form described above, then, due to (6.2), we have that $|[A, A]| = 0$.

On the other hand if $|[A, A]| = 0$, then due to Newlander-Nirenberg theorem $[NN]$, we have that there exist local complex coordinates, say, $z_1 = s_1 + it_1$ and $z_2 = s_2 + it_2$, in some neighbourhood of $a \in M$, such that:

$$||A|| = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

in the basis $\left(\frac{\partial}{\partial s_1}, \frac{\partial}{\partial t_1}, \frac{\partial}{\partial s_2}, \frac{\partial}{\partial t_2}\right)$.

If we do the following coordinate transformation:

$$(s_1, t_1, s_2, t_2,) \longmapsto (s_1, t_1, t_2, s_2),$$

we transform the standard structure on $\mathbb{C}^2$ to the almost complex structure (6.2) of

Π. ■

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