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Non-existence of higher-dimensional pseudoholomorphic submanifolds

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Abstract. In this paper we prove that only pseudoholomorphic curves appear as J-invariant submanifolds of generic almost complex manifolds (M, J). We also prove there exist no non-trivial automorphisms or submersions of such manifolds. On the other hand we show that abundance of 1-jets of PH-submanifolds, automorphisms or submersions implies integrability of the almost complex structure.

Introduction

Let a manifold M^{2m} be equipped with an almost complex structure $J \in \text{End}(TM)$, $J^2 = -\mathbf{1}$. A submanifold $L \subset M$ is called *pseudoholomorphic* (PH-submanifold) if $TL \subset TM$ is J-invariant. Existence of local PH-submanifolds of complex dimension one was first proved by Nijenhuis and Woolf [NW]. The theory has been revolutionized by the paper of Gromov [Gr], where many global results for PH-curves were established and applied to symplectic geometry (see further results in [MS]).

It was believed there are no higher dimensional PH-submanifolds for generic J. This was stated without proof by Gromov [Gr] and even more indirectly by Donaldson [D] ("one does not expect to find any solutions"). However though an overdetermined system with generic coefficients is usually non-integrable, it may be solvable. This paper is devoted to a clarification of the question.

Denote by $\mathcal{J}(M)$ the space of almost complex structures on a manifold M^{2m}.

Theorem. *There exists an open dense in C^r-topology subset $\mathcal{J}' \subset \mathcal{J}$ such that an almost complex manifold (M, J) with $J \in \mathcal{J}'$ has no local (even formal)*

- *PH-submanifolds of dimension $2n$, $2 \le n \le m-1$. Here $r = \max\{2, 6-n\}$.*
- *PH-submanifolds of dimension $2n$, $2 \le n \le m-1$, through a generic point. Here $r = \max\{1, 5-n\}$.*
- *PH-automorphisms $f \in \text{Aut}_{loc}(M, J)$ different from id_M. In this case $r = \max\{1, 5-m\}$.*

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– *local PH-submersions onto an almost complex manifold of dimension* $2n$*,* $0 < n < m$*. Here* $r = 2$ *for* $(m, n) = (2, 1)$ *and* $r = 1$ *else.*

To prove the claim we consider the set of N_J-invariant $2n$-planes and show that it contains generically no integrable sub-distributions. The real analog for $n = 2$ is integrable in the C^ω-category: if A is a (2, 1)-tensor on M^m, the equation $A(TL, TL) \subset TL$ on a surface $L^2 \subset M$ is determined and is in Cauchy-Kovalevskaya form. In the presence of J we get an overdetermined equation and we should find enough compatibility conditions to obstruct any solution.

A similar statement takes place in Riemannian geometry too, where one should change PH-maps to isometries and PH-submanifolds $L^{2n} \subset (M^{2m}, J)$ to totally geodesic $L^n \subset (M^m, g)$, $n > 1$. The arguments are similar. However the author has found neither a proof nor even the statement in the literature.

An almost complex manifold (M, J) has no more local PH-submanifolds (usually none) than a complex one, with the only exception of PH-curves. Moreover if it has as many PH-submanifolds as in the integrable situation, then the structure J is actually integrable. See the precise statement and the comparison to an analogous result of McKay [M] in the conclusion (Theorem 15 in §3.1).

The non-existence theorem for PH-submanifolds suggests examining some other analogs, which were actually established in [D]. Namely Donaldson introduced and applied the approximate PH-submanifolds. At the end of the paper we give another approach to his concept, arising from quantization theories.

1. Linear approximation

1.1. Preliminaries on the Nijenhuis tensor

The Nijenhuis tensor of an almost complex structure J is given by the following formula:

$$N_J(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

This is a skew-symmetric and J-antilinear in each argument (2, 1)-tensor.

Definition 1. *Define the space of linear Nijenhuis tensors on a complex vector space* (V, J) *by*

$$\mathcal{N}(V, J) = \{N \in \Lambda^2 V^* \otimes V \mid N(JX, Y) = -JN(X, Y)\}.$$

Due to [Kr1] every such a linear tensor on $V = T_x M$ can be realized as the Nijenhuis tensor of some almost complex structure in a neighborhood of $x \in M$.

Definition 2. *For a* (2, 1)*-tensor* N *on a vector space* V *a subspace* $W \subset V$ *is called* N*-invariant if* $N(W, W) \subset W$*.*

Proposition 1. *Let* $L \subset (M, J)$ *be an almost complex submanifold, i.e.* $T_x L$ *is* J*-invariant for every* $x \in L$*. Then every* $T_x L$ *is also* N_J*-invariant.* □

Motivated by this obvious characterization of tangent spaces to PH-submanifolds we introduce the following notion.

Definition 3. *Let $N \in \mathcal{N}(V, J)$. The N-Grassmannian of type $2n$ is:*

$$\begin{aligned}\mathrm{Gr}_{2n}(V, J, N) &= \{\Pi^{2n} \subset V \mid J\Pi = \Pi,\ N(\Pi, \Pi) \subset \Pi\}\\ &\subset\ \mathrm{Gr}_n^{\mathbb{C}}(V) = \mathrm{Gr}_{2n}(V, J).\end{aligned}$$

Thus to each almost complex manifold (M, J) we can associate the "bundle" $(M, J)\,\mathrm{Gr}_{2n}(M, J, N_J) = \cup_x \mathrm{Gr}_{2n}(T_x M, J, N_J)$ (the "fibers" over different $x \in M$ could be non-isomorphic).

Example. Consider the unit sphere $S^6 \subset \mathbb{R}^7$. Its well-known almost complex structure J is defined as follows: Identify $\mathbb{R}^7$ with the purely imaginary octonions. The imaginary part of the octonion multiplication gives the vector product $\times$ on $\mathbb{R}^7$. We define $J : T_x S^6 \to T_x S^6$ by $\eta \mapsto x \times \eta$, where $\eta \in \mathbb{R}^7$ and $\eta \perp x$.

Because J on S^6 is homogeneous, the tensor N_J is the same at all points. Identify the octonions with $\mathbb{H}[1, l]$, where $\mathbb{H}$ is the space of quaternions $\mathbb{R}^4 \simeq \mathbb{R}[1, i, j, k]$, $k = ij$. Choose the point $x = i \in S^6$. The tangent plane is $T_x S^6 = \langle j, k, l, il, jl, kl\rangle$. Let us fix a real 3-space $L^3 = \langle X_1, X_2, X_3\rangle$ with $X_1 = \frac{1}{2}k$, $X_2 = \frac{1}{2}il$, $X_3 = \frac{1}{2}kl$. Then the Nijenhuis tensor is given by the following table:

$N_J(\uparrow, \leftarrow)$	X_1	X_2	X_3	JX_1	JX_2	JX_3
X_1	0	X_3	$-X_2$	0	$-JX_3$	JX_2
X_2	$-X_3$	0	X_1	JX_3	0	$-JX_1$
X_3	X_2	$-X_1$	0	$-JX_2$	JX_1	0
JX_1	0	$-JX_3$	JX_2	0	$-X_3$	X_2
JX_2	JX_3	0	$-JX_1$	X_3	0	$-X_1$
JX_3	$-JX_2$	JX_1	0	$-X_2$	X_1	0

Note that N_J-multiplication coincides with the standard vector product on $L^3 \simeq \mathbb{R}^3$. Thus $\mathrm{Gr}_4(S^6, J, N_J) = \emptyset$ and (S^6, J) contains no 4-dimensional PH-submanifolds. However there are plenty of PH-spheres $S^2 \subset S^6$.

Remark. The example we considered is exceptional for two reasons. First, the Nijenhuis tensor is obtained as the complexification of some real (2, 1)-tensor on a totally real subspace L: $T_x M = L \otimes \mathbb{C}$ and N is extended by anti-linearity. The tensors possessing such a realification are exceptional for $m > 2$. And second, as we shall see, under certain restrictions the N_J-Grassmannian is not empty.

1.2. Study of the N-Grassmannians

Proposition 2. *For generic $N \in \mathcal{N}(V, J)$ the Grassmannian $\mathrm{Gr}_{2n}(V, J, N) \subset \mathrm{Gr}_n^{\mathbb{C}}(V)$ is a smooth submanifold of dimension $2(m-2)$ for $n = 2$, is discrete for $n = 3$ and is empty for $n \geq 4$.*

Proof. Let π be the bundle over $\mathrm{Gr}_n^{\mathbb{C}}(V)$ with the fiber $\pi^{-1}(\Pi)$ at $\Pi \in \mathrm{Gr}_n^{\mathbb{C}}(V)$ equal $\mathrm{Hom}_{\bar{\mathbb{C}}}(\Lambda^2_{\mathbb{C}}\Pi, \nu)$. Here $\mathrm{Hom}_{\bar{\mathbb{C}}}$ denotes the set of anti-holomorphic morphisms, $\nu = V/\Pi$ is the normal bundle and $\Lambda^2_{\mathbb{C}}\Pi$ is the exterior power over $\mathbb{C}$, i.e. the quotient of $\Lambda^2\Pi$ by the equivalence $JX \wedge Y = X \wedge JY$. If X_i $(1 \le i \le n)$ is a $\mathbb{C}$-basis of Π, then $X_i \wedge X_j$ is a $\mathbb{C}$-basis of $\Lambda^2_{\mathbb{C}}\Pi$. Define the section $\Gamma_N \in C^\infty(\pi)$ in the following way:

$$\Gamma_N(\Pi)(X_i \wedge X_j) = N(X_i, X_j) \bmod \Pi, \quad \Pi \in \mathrm{Gr}_n^{\mathbb{C}}(V).$$

Lemma 3. *For generic $N \in \mathcal{N}(V, J)$ the canonical section $\Gamma_N \in C^\infty(\pi)$ is transversal to the zero section.*

Proof. We will actually prove more, namely that the set of non-generic N is a stratified submanifold of the vector space $\mathcal{N}(V, J)$ of positive codimension. Since $\mathrm{Gr}_n^{\mathbb{C}}(V)$ has a finite atlas, it suffices to prove the statement in a chart.

Let $V = \Pi \oplus \nu$ be a complex decomposition with bases $X_1, \dots, X_n$ and $X_{n+1}, \dots, X_m$ respectively. It produces the chart $U(\Pi, \nu) \subset \mathrm{Gr}_n^{\mathbb{C}}(V)$ represented by elements $\sigma \in \mathrm{Hom}_{\mathbb{C}}(\Pi, \nu)$. Let $\sigma(X_i) = \sum_{j=n+1}^m b_i^j X_j$. Then we associate the subspace $L_\sigma = \mathrm{graph}(\sigma) = \mathbb{C}\langle \xi_1, \dots, \xi_n\rangle$ to σ, where $\xi_i = X_i + \sigma(X_i)$.

Let us write the above direct sum as $w = w_\Pi + w_\nu$. The projection $P_\sigma : V \to \nu$ along L_σ is given by the formula $P_\sigma(w) = w_\nu - \sigma(w_\Pi)$. So if we specify the Nijenhuis tensor by $N(X_i, X_j) = \sum_{r=1}^m a_{ij}^r X_r$, $1 \le i < j \le n$, the canonical section of π is given by

$$\Gamma_N : \xi_i \wedge \xi_j \mapsto \bigl[(a_{ij}^r + \bar{b}_i^s a_{sj}^r + \bar{b}_j^t a_{it}^r + \bar{b}_i^s \bar{b}_j^t a_{st}^r) - b_q^r (a_{ij}^q + \bar{b}_i^s a_{sj}^q + \bar{b}_j^t a_{it}^q + \bar{b}_i^s \bar{b}_j^t a_{st}^q)\bigr] X_r.$$

Here we assume summation by repeated $n + 1 \le r, s, t \le m$ and $1 \le q \le n$.

Intersection of Γ_N with the zero section is given by the cubic system $c(a, b)_{ij}^r = 0$. The free term a_{ij}^r does not enter other places, so that $\gamma(a, b)_{ij}^r := a_{ij}^r - c(a, b)_{ij}^r$ does not depend on it. By the Sard theorem generic a_{ij}^r is a regular value of $\gamma(a, b)_{ij}^r$, so that for the corresponding tensor N we get the required transversality. □

Now we finish the proof of the proposition. Let N be generic as in the lemma. The rank of the bundle π is $n(n-1)(m-n)$ and the dimension of the base is $2n(m-n)$, whence the claim (a generic section has no zeros if the fiber dimension is larger than that of the base). □

Remark. The following modification of the proof was suggested by the referee. One pulls back the bundle π to $\mathcal{N}(V, J) \times \mathrm{Gr}_n^{\mathbb{C}}(V)$. The new bundle $\hat{\pi}$ has the canonical section $\hat{\Gamma} : (N, \Pi) \mapsto \Gamma_N(\Pi)$. By the calculation in a chart this section is transversal to the zero section. Thus we obtain the real algebraic set $\hat{\Gamma}^{-1}(0) = \sqcup_{N \in \mathcal{N}(V,J)} \mathrm{Gr}_{2n}(V, J, N)$. The conclusion follows by application of the Sard theorem to the projection of $\hat{\Gamma}^{-1}(0)$ to $\mathcal{N}(V, J)$.

So for $n \ge 3$ the $\mathrm{Gr}_{2n}(TM, J, N_J)$ is locally a finite collection of rank $2n$ distributions. Integrability of them is an additional restriction on N_J, which is not fulfilled for a generic J. On the other hand the case $n = 2$ provides families of sections and it is not clear why there's none integrable among them.

1.3. Complex linear algebra

To understand N-Grassmannians let us study the operators $N(X,\cdot)$, $X \in V$.

Proposition 4. *Let* (V, J) *be a complex linear space,* $J^2 = -\mathbf{1}$, *and let* $A : V \to V$ *be a* J*-antilinear operator. Then there exists a decomposition* $V = V_1 \oplus V_2$ *such that* $AV_i \subset V_i$ *and* $JV_i = V_{i+1}$, $i \in \mathbb{Z}\,(\mathrm{mod}\,2)$.

Proof. Spectrum of the operator is $\mathrm{Sp}(A) = \Lambda_+ \cup \Lambda_- \cup \Lambda_0$, where $\Lambda_\varepsilon = \{\lambda \in \mathrm{Sp}(A) \mid \mathrm{sgn}(\mathrm{Re}\,\lambda) = \varepsilon\}$. Let $V = V_+ \oplus V_- \oplus V_0$ be the corresponding splitting. Then $JV_\pm = V_\mp$ and $JV_0 = V_0$. So it is enough to consider $\Lambda_0 = \{0, \pm i\lambda_1, \dots, \pm i\lambda_s\}$. We consider only one summand of the resulting splitting.

Consider at first $\lambda_j \neq 0$. Let $\hat{V}$ be an A, J-invariant subspace with $\mathrm{Sp}(A) = \{\pm i\lambda_j\}$. Suppose A is semisimple. Denote $I = \lambda_j^{-1}A$ and $K = IJ$. Then one easily checks that the operators $(\mathbf{1}, I, J, K)$ determine a quaternion representation on $\hat{V}$. Thus $\hat{V} = \oplus \hat{V}_l$, each summand being invariant and of dimension 4. Choosing $\xi \in \hat{V}_l \setminus \{0\}$ we get a decomposition $\hat{V}_l = \langle \xi, I\xi\rangle \oplus \langle J\xi, K\xi\rangle$, whence the claim. If A is not semisimple, we get the Jordan filtration $\hat{V}_0 \subset \hat{V}_1 \subset \dots \subset \hat{V}$ and the proof is achieved similarly with the adjoint grading.

For the zero eigenvalue $\lambda = 0$ the operator A vanishes on the lowest filtration term $\hat{V}_0$, which can be decomposed as specified. Thus by climbing the Jordan tower we successively construct the required decomposition. □

Corollary 5. *If* $\dim V \in 4\mathbb{Z} + 2$, *then* A *has an invariant line in every summand* $V_{1,2}$. *Therefore* V *contains a 2-dimensional plane that is* A, J*-invariant.* □

Corollary 6. *Let* $N \in \mathcal{N}(V, J)$ *be some linear Nijenhuis tensor. If* $\dim V \in 4\mathbb{Z}$, *then there exists a* J, N*-invariant 4-dimensional subspace* $W \subset V$.

Proof. Take $X \in V \setminus \{0\}$. Let $L^2 = \langle X, JX\rangle$ and Π be a J-invariant complement, $V = \Pi \oplus L^2$. Denote by $\pi : V \to \Pi$ the projection along L and define $A : \Pi \to \Pi$ as the composition $\pi \circ N(X,\cdot)$. $N \in \mathcal{N}(V, J)$ implies $AJ + JA = 0$. So the claim follows from the previous corollary. □

Thus for even m the "bundle" $\mathrm{Gr}_4(M, J, N_J)$ is always non-empty.

1.4. Classification of low-dimensional Nijenhuis tensors

If $m = \dim_{\mathbb{C}} V = 2$, any non-zero linear Nijenhuis tensor is given by relations $N(zX_1, wX_2) = \bar{z}\bar{w}X_1$, where $(x + iy)X = xX + yJX$.

We present a classification in the case $m = 3$. Let us call a Nijenhuis tensor N *non-degenerate* if $N(X, Y) = 0$ implies $\mathbb{C}\langle X\rangle = \mathbb{C}\langle Y\rangle$. Note that in this case $N : \Lambda^2_{\mathbb{C}} V \to V$ is an anti-isomorphism of 3-dimensional complex spaces. Since every element of $\Lambda^2_{\mathbb{C}} V$ is represented by a decomposable bivector, the image $\Pi = N(\Lambda^2_{\mathbb{C}} V)$ is a complex vector subspace of V. Denote the non-degenerate case by NDG and the case when $\dim \mathrm{Im}(N) = 4$ (resp. 2) by DG_1 (resp. DG_2).

In the statement below we present N via some complex basis of $V \simeq \mathbb{C}^n$ using the anti-linear rule $N(zX, wY) = \bar{z}\bar{w}N(X, Y)$. Variables $\lambda > 0$, φ, ψ are supposed to be real.

Theorem 7. *Any linear non-zero Nijenhuis tensor N can be carried to one of the forms:*

NDG: *1.* $N(X_1, X_2) = X_2$, $N(X_1, X_3) = \lambda X_3$, $N(X_2, X_3) = e^{i\varphi}X_1$,
2. $N(X_1, X_2) = X_2$, $N(X_1, X_3) = X_3 + X_2$, $N(X_2, X_3) = e^{i\varphi}X_1$,
3. $N(X_1, X_2) = \cos\psi X_2 + \sin\psi X_3$,
$N(X_1, X_3) = -\sin\psi X_2 + \cos\psi X_3$, $N(X_2, X_3) = e^{i\varphi}X_1$,
4. $N(X_1, X_2) = X_1$, $N(X_1, X_3) = X_2$, $N(X_2, X_3) = X_2 + X_3$,

DG$_1$: *1.* $N(X_1, X_2) = X_2$, $N(X_1, X_3) = \lambda X_3$, $N(X_2, X_3) = 0$,
2. $N(X_1, X_2) = X_2$, $N(X_1, X_3) = X_3 + X_2$, $N(X_2, X_3) = 0$,
3. $N(X_1, X_2) = \cos\psi X_2 + \sin\psi X_3$,
$N(X_1, X_3) = -\sin\psi X_2 + \cos\psi X_3$, $N(X_2, X_3) = 0$,
4. $N(X_1, X_2) = X_1$, $N(X_1, X_3) = X_2$, $N(X_2, X_3) = 0$,
5. $N(X_1, X_2) = X_2$, $N(X_1, X_3) = X_1$, $N(X_2, X_3) = 0$.

DG$_2$: *1.* $N(X_1, X_2) = X_1$, $N(X_1, X_3) = 0$, $N(X_2, X_3) = 0$,
2. $N(X_1, X_2) = X_3$, $N(X_1, X_3) = 0$, $N(X_2, X_3) = 0$.

The above forms are pairwise non-equivalent save for some exceptional values of parameters.

Proof. NDG. Consider the map $\Phi_1 : \mathbb{C}P^2 \to \mathrm{Gr}_2^{\mathbb{C}}(3) \simeq \mathbb{C}P^2$ given by $\mathbb{C}\langle X\rangle \mapsto \mathrm{Im}\, N(X, \cdot)$. Let $\Phi_2 : \mathrm{Gr}_2^{\mathbb{C}}(3) \to \mathbb{C}P^2$ be the mapping $\mathbb{C}^2\langle Y, Z\rangle \mapsto \mathbb{C}\langle N(Y, Z)\rangle$. By the non-degeneracy assumption both are correctly defined and are diffeomorphisms. So $\Phi = \Phi_2 \circ \Phi_1$ is a diffeomorphism of $\mathbb{C}P^2$.

Its Lefschetz number is $l(\Phi) = t_0 + t_2 + t_4$, where

$$t_i = \mathrm{tr}(\Phi^*) : H^i_{dR}(\mathbb{C}P^2) \to H^i_{dR}(\mathbb{C}P^2).$$

Thus $t_0 = 1$, $t_2 = \pm 1$ and $t_4 = 1$ since Φ preserves orientation. So $l(\Phi) \neq 0$ and there is a fixed point, i.e. for some $\mathbb{C}$-invariant subspaces $L^2, \Pi^4 \subset V^6$ we have $\Phi_1(L^2) = \Pi^4$, $\Phi_2(\Pi^4) = L^2$. We choose $L^2 = \mathbb{C}\langle X_1\rangle$.

There are two possibilities: either $L^2 \cap \Pi^4 = \{0\}$ or $L^2 \subset \Pi^4$. In the first case we study the operator $N_{X_1} \stackrel{\text{def}}{=} N(X_1, \cdot) : \Pi^4 \to \Pi^4$. We note that since N_{X_1} is real, the spectrum $\mathrm{Sp}(N_{X_1})$ is conjugacy-invariant. But due to anti-linearity of N_{X_1}, it is also invariant under multiplication by -1. Moreover if a two-dimensional space in Π^4 is J, N_{X_1}-invariant, it corresponds to real eigenvalues $\{\pm\lambda\}$ (because N_{X_1} is orientation-reversing on this 2-plane). Finally $\mathbb{R}$-scaling of X_1 results in real-scaling of $\mathrm{Sp}(N_{X_1})$ and S^1-scaling preserves the spectrum. Thus by Proposition 4 we get the normal forms (1–3).

The second possibility corresponds to N-invariant space Π^4, so we use classification in dimension $2m = 4$ and choose a transversal vector X_3: $N(X_1, X_2) = X_1$, $N(X_1, X_3) = X_2$, $N(X_2, X_3) = aX_3 + bX_2 + cX_1$. The transformation $\tilde{X}_1 = \sigma X_1$, $\tilde{X}_2 = \frac{\bar{\sigma}}{\sigma}X_2 + \kappa X_1$, $\tilde{X}_3 = \frac{1}{\bar{\sigma}}X_3 + \frac{\bar{\kappa}}{\sigma}X_2 + \mu X_1$ changes coefficients: $\tilde{a} = a$, $\tilde{b} = \frac{1}{\bar{\sigma}}[\bar{\kappa}(1-a) + \frac{\sigma}{\bar{\sigma}}b]$, $\tilde{c} = |\sigma|^{-2}[|\kappa|^2 a - \kappa\frac{\sigma}{\bar{\sigma}}b + c - (\mu\bar{\sigma}a + \bar{\mu}\sigma)]$. Thus $|a| \neq 1$

reduces to $b = c = 0$ and so is the form (1). The case $a = e^{2i\varphi}$, $0 < \varphi < \pi$, reduces to $b = 0$, $c = i\lambda e^{i\varphi}$, with $\lambda = 0, 1$, which corresponds to the forms (1) and (2) respectively. Finally the case $a = 1$ reduces to $b = 0, 1$, $c = 0$ and for $b = 1$ we get the new form (4).

DG_1. We suppose $N(X_2, X_3) = 0$ for $\mathbb{C}$-independent X_2, X_3. Then denote $\Pi^4 = \mathbb{C}^2\langle N(X_1, X_2), N(X_1, X_3)\rangle$. If $N(\Pi^4, \Pi^4) = 0$, then $\Pi^4 = \mathbb{C}^2\langle X_2, X_3\rangle$ and we study the operator $N(X_1, \cdot) : \Pi^4 \to \Pi^4$ to get forms (1–3). Otherwise $N(\Pi^4, \Pi^4) = L^2$ and there are two possibilities. If $L^2 \cap \mathbb{C}^2\langle X_2, X_3\rangle = \{0\}$ we choose $L^2 = \mathbb{C}\langle X_1\rangle$, $\langle X_2, X_3\rangle \cap \Pi^4 = \mathbb{C}\langle X_2\rangle$ and get the case (4). And if $L^2 \subset \mathbb{C}^2\langle X_2, X_3\rangle$ we choose $X_2 \in L^2$, $X_1 \in \Pi^4 \setminus L^2$ and get the form (5).

DG_2. Here $\Pi^2 = \mathrm{Im}\, N \subset V^6$. We define $X_3 \in L^2 = \mathrm{Ker}\, N \subset V^6$. Then the two forms (1–2) correspond to the cases $L^2 \cap \Pi^2 = \{0\}$ and $L^2 = \Pi^2$.

Now degenerate cases are obviously pairwise non-isomorphic. So we prove non-equivalence of the cases NDG(1-4). In case (1) with $\lambda \neq 1$ the map Φ has 3 fixed points $L^2 \subset V$ on $\mathbb{C}P^2$, but 2 of them satisfy $L^2 \subset \Phi_1(L^2)$. In case (2) provided $\varphi \neq \pm\frac{\pi}{2}$ there are 2 fixed points (one degenerate). For (3) with $\psi \neq \pm\frac{\pi}{2} \pm \varphi$ or $\psi \neq \frac{\pi k}{2}$ we have 3 fixed points, but none satisfies $L^2 \subset \Phi_1(L^2)$. In the last case (4) Φ has a unique fixed point. For all the exceptional cases indicated above the number of fixed points is infinite. □

We show one adjacent result. Fix a type of the Nijenhuis tensor NDG(1–4) with varying parameters. Call it non-exceptional if no parameter is exceptional.

Proposition 8. *S^6 has no almost complex structure of non-exceptional type.*

Proof. Otherwise TS^6 has a proper subbundle – a fixed point of Φ. □

The standard structure (§1.1) is of exceptional type: $N_J \in \mathrm{NDG}(3)_{\varphi=0,\psi=\frac{\pi}{2}}$.

1.5. Structure of $\mathrm{Gr}_4(V, J, N)$

In this subsection we re-prove Proposition 2 for $n = 2$ and consider in details the case $m = 3$.

Proposition 9. *For a generic tensor $N \in \mathcal{N}(V, J)$, the set $\mathrm{Gr}_4(V, J, N)$ is a stratified submanifold of $\mathrm{Gr}_2^{\mathbb{C}}(V)$ of real dimension $2(m-2)$.*

Proof. Note that $\mathrm{Gr}_4(V, J, N) = \mathrm{Gr}_4^0 \cup \mathrm{Gr}_4^1$, where

$$\mathrm{Gr}_4^k = \{\Pi^4 \in \mathrm{Gr}_4(V, J, N) \mid \dim_{\mathbb{C}} N(\Lambda^2\Pi) = k\}.$$

Denote the linear map $N(X, \cdot) : V \to V$ by N_X and its characteristic polynomial by $P_X(\lambda) = \det(N_X - \lambda\mathbf{1})$. Since $N_X(\mathbb{C}\langle X\rangle) = 0$, we have $P_X(0) = P'_X(0) = 0$. The condition $X \in \Pi \in \mathrm{Gr}_4^0$ means $\exists Y \notin \mathbb{C}\langle X\rangle$ such that $N_X(\mathbb{C}\langle Y\rangle) = 0$. Thus for such X we have $P''_X(0) = P'''_X(0) = 0$ and these equations define a stratified submanifold in $\mathbb{C}P(V)$ of dim $\leq 2(m-2)$. Moreover rotation in the plane $\mathbb{C}^2\langle X, Y\rangle$ reduces dimension and so $\mathrm{Gr}_4^0 \subset \mathrm{Gr}_2^{\mathbb{C}}(V)$ is a stratified submanifold of real dimension $\leq 2(m-3)$.

If $\Pi^4 \in \mathrm{Gr}_4^1$, $L^2 = N(\Pi, \Pi)$ and $X \in \Pi \setminus L^2$, then $N_X : L^2 \to L^2$ and eigenvalues are non-zero real numbers $\{\pm\lambda\}$. Let

$$\mathcal{U} = \bigsqcup_{k=1}^{m-1} \mathcal{U}_k,\ \mathcal{U}_k = \{\mathbb{C}\langle X\rangle \mid \exists\, k \text{ blocks of real eigenvalues } \{\pm\lambda \neq 0\} \text{ in } \mathrm{Sp}(N_X)\}.$$

Decompose $\mathcal{U}_k = \mathcal{U}_k' \cup \mathcal{U}_k''$ with $\mathcal{U}_k''$ corresponding to multiple eigenvalues. For a generic tensor N the set $\mathcal{U}_k'$ is open and dense in $\mathcal{U}_k$. Each $\mathbb{C}\langle X\rangle \in \mathcal{U}_k'$ determines exactly k complex planes $\mathbb{C}^2\langle X, Y\rangle$, where Y is an eigenvector corresponding to a real eigenvalue of N_X (there are families over points of $\mathcal{U}_k''$ of dimension equal to the codimension of the corresponding $\mathcal{U}_k''$-stratum in $\mathcal{U}_k$).

Generically the set $\mathcal{U}$ is open in $\mathbb{C}P(V)$ with $\dim_{\mathbb{C}} \mathcal{U} = m - 1$. Since $\mathbb{C}\langle X\rangle$ is defined up to transformation $X \mapsto X + aY$, $a \in \mathbb{C}$, the statement is proved. □

Proposition 10. *For $m = 3$ and generic N the Grassmannian $\mathrm{Gr}_4(V, J, N)$ is one of the manifolds $\emptyset$, S^2, T^2.*

Proof. Let's identify $V \simeq \mathbb{C}^3$. We define a complex linear map $\times : \Lambda^2_{\mathbb{C}} V \to V$ as the complexification of the standard vector product on $\mathbb{R}^3$. Using this isomorphism we define a complex map $\hat{N} : V \to V$ as the composition of $N \circ (\times^{-1})$ and the conjugation. Now the condition $\Pi^4 = \times^{-1}(\mathbb{C}\langle X\rangle) \in \mathrm{Gr}_4(V, J, N)$ reads $(X, \hat{N}X) = 0$. This is a real quadric in $\mathbb{C}P^2$ of codimension 2. We assume the identification as well as the conjugation and the Hermitian metric are given by the basis from Proposition 7 and use in calculations those normal forms.

NDG(1). Suppose $\lambda \neq \pm 1$. If $\Pi^4 = \mathbb{C}^2\langle X_1 + zX_3, X_2 + wX_3\rangle \in \mathrm{Gr}_4(V, J, N)$, we have $w = |z|^2 \left(\dfrac{\cos\varphi}{1-\lambda} + i \dfrac{\sin\varphi}{1+\lambda} \right)$. At infinity $\mathbb{C}P^1 \subset \mathbb{C}P^2$ only one point $\mathbb{C}^2\langle X_1, X_3\rangle$ is added and so $\mathrm{Gr}_4(V, J, N) \simeq S^2$.

NDG(2). In this case with the above notations $|w|^2 = |z|^2 \cos\varphi$, $\mathrm{Im}\, w = \frac{1}{2}|z|^2 \sin\varphi$. Let $\cos\varphi \sin\varphi \neq 0$. For $\cos\varphi < 0$ the Grassmannian is empty. For $\cos\varphi > 0$ it corresponds to the graph over a domain $|z| \leq \dfrac{2\sqrt{\cos\varphi}}{|\sin\varphi|}$ given by $w = \frac{1}{2}|z| \left(\pm\sqrt{4\cos\varphi - |z|^2 \sin^2\varphi} + i|z|\sin\varphi \right)$. Thus it is an immersed $S^2 \subset \mathbb{C}P^2$ with one singularity at $(z, w) = (0, 0)$ of the type of the standard cone in $\mathbb{R}^3$. Notice though that this singularity does not contradict the statement of Proposition 2, because the Jordan box case is not generic.

NDG(3). The corresponding equation $|z|^2 e^{i\varphi} = \cos\psi\,(w - \bar{w}) - \sin\psi\,(1 + |w|^2)$ has no solutions if $\cos\varphi / \sin\psi > 0$ or $|\cos\varphi| < |\sin\psi|$. For the opposite inequalities the Grassmannian $\mathrm{Gr}_4(V, J, N)$ is a torus T^2 projected to the annulus

$$\rho_1 \leq |z|^2 \leq \rho_2, \quad \text{where} \quad \rho_{1,2} = -\frac{2\cos\varphi\cos^2\psi}{\sin^2\varphi\sin\psi} \left(1 \mp \sqrt{1 - \tan^2\varphi\tan^2\psi} \right).$$

NDG(4). The Grassmannian is defined by the equation $2\,\mathrm{Re}\, z + |w|^2 = w\bar{z}$. The solution $2w = z(1 \pm \sqrt{|z-4|^2 - 4^2}/|z|)$ is defined outside the disk $B_2(4) \subset \mathbb{C}$ and has a one point compactification at infinity. So we get S^2. □

For non-generic Nijenhuis tensors, the dimension of the Grassmannian can increase. For example for N of the type NDG(1) with the parameter $\lambda = 1$ the dimension of $\mathrm{Gr}_4(V, J, N)$ is 3. But it is always less than 4 if $N \neq 0$.

2. Differential equations approach

2.1. *General scheme of PDEs investigation*

Here we briefly review the geometric theory of (systems of) PDEs. The reader is asked to consult [KLV], [Sp], [Gu] and [Ly] for details.

From the geometric point of view a PDE of order k is a submanifold $\mathcal{E}_k$ in the jet space $J^k(\pi)$ (of some bundle π) equipped with the canonical Cartan distribution $\mathcal{C}_k$. The last is defined as follows: Let $\pi_{k,k-1} : \mathcal{E}_k \to \mathcal{E}_{k-1}$ be the forgetful projection (one can assume $\mathcal{E}_{k-1} = J^{k-1}(\pi)$ for simplicity). If $x_k = [s]^k_x$ is the k-jet of a section s at a point x, we have $\pi_{k,k-1}(x_k) = x_{k-1} = [s]^{k-1}_x$. Denote by $L(x_k) = T_{x_{k-1}} j_{k-1}(s)$ the tangent space to the jet section. Then we define $U_k : T_{x_k}\mathcal{E}_k \stackrel{(\pi_{k,k-1})_*}{\longrightarrow} T_{x_{k-1}}\mathcal{E}_{k-1} \to T_{x_{k-1}}\mathcal{E}_{k-1}/L(x_k)$ and $\mathcal{C}_k = \mathrm{Ker}\, U_k$.

At every point $x_k \in \mathcal{E}_k$ the Cartan subspace can be decomposed $\mathcal{C}_k = H \oplus g_k$, where $g_k(x_k) = \mathrm{Ker}(\pi_{k,k-1})_*$ is the symbol of $\mathcal{E}_k$ and H is some additional horizontal subspace that projects isomorphically by $\pi_k : \mathcal{E}_k \to M$ to $T_x M$.

We consider equations modelled on the trivial bundle $\pi : M \times N \to M$. In this case the symbol $g_k \subset S^k\tau^* \otimes \upsilon$, where $\tau = T_x M$, $\upsilon = T_y N$ and $(x, y) = \pi_{k,0}(x_k)$. Differentiation of the equations corresponds to the prolongation of the system $\mathcal{E}_{k+1} = \mathcal{E}_k^{(1)} \subset J^{k+1}(\pi)$. Prolongation over a point $x_k \in \mathcal{E}_k$ exists if it belongs to the image of the projection $\pi_{k+1,k}(\mathcal{E}_{k+1})$. A PDE $\mathcal{E}$ all the points of which have infinite prolongations and the fibers behave regularly is called formally integrable.

The machinery to study prolongations is the Spencer δ-cohomology. The algebraic prolongations of the symbol are defined inductively as $g_{l+1} = g_l^{(1)} = \mathrm{Ker}(\delta : g_l \otimes \tau^* \to g_{l-1} \otimes \Lambda^2\tau^*)$, $l \geq k$, where the operator δ is the symbol of the de Rham differential. They can be also calculated by the formula $g_l = g_k^{(l-k)} = S^{l-k}\tau^* \otimes g_k \cap S^l\tau^* \otimes \upsilon$. Spencer group $H^{i,j}(g)$ is the cohomology of the following δ-complex:

$$\cdots \to g_{i+1} \otimes \Lambda^{j-1}\tau^* \stackrel{\delta}{\to} g_i \otimes \Lambda^j\tau^* \stackrel{\delta}{\to} g_{i-1} \otimes \Lambda^{j+1}\tau^* \to \ldots$$

It was known since Quillen and Goldschmidt that the groups $H^{i,2}$ obstruct to prolongations ([Sp]). Obstructions $W_k(x_k) \in H^{k-1,2}(\mathcal{E}_k, x_k)$, introduced in [Ly] (see also [KL]), are called Weyl tensors and have the following definition.

Let $\Omega_k = dU_k|_{\mathrm{Ker}\, U_k} : \Lambda^2\mathcal{C}_k \to g_{k-1}$ be the metasymplectic structure. Its restriction to a horizontal subspace $\Omega_k|_H \in \Lambda^2 H^* \otimes g_{k-1} \simeq g_{k-1} \otimes \Lambda^2\tau^*$ is δ-closed. Another choice of horizontal space $H \subset \mathcal{C}_k$ results in a change of $\Omega_k|_H$ by a δ-exact form. So the δ-cohomology class $W_k = [\Omega_k|_H]$ is well-defined. If $\mathcal{E}$ is a defining equation for a G-structure, then one can show that the Weyl tensor is the well-known structure function ([Gu], [St]).

The set of once-prolongable points $x_k \in \mathcal{E}_k$ has the equation $W_k(x_k) = 0$. If the equation holds identically we prolong once and study $\mathcal{E}_{k+1}$. Otherwise we get a new equation $\tilde{\mathcal{E}}_k \subset \mathcal{E}_k$ determined by the condition $W_k = 0$. We apply the machinery to this equation and so on. Moreover it can happen that the projection $\pi_{k,k-1}$ restricted to $\tilde{\mathcal{E}}_k$ is not epi. Then we get a system of smaller order, which we study etc.

Thus we obtain prolongation-projection method for our investigation. The Cartan-Kuranishi theorem says the procedure stops in a finite number of steps (for regular points). Using a well-known phrase of E. Cartan "after a finite number of prolongations the system becomes involutive or contradicting". We will apply this technique to get a contradiction (non-existence) in two steps.

Remark. An alternative approach to PDEs is the Cartan's theory of exterior differential systems ([C]). See the modern exposition in [BCG].

2.2. Equation on PH-submanifolds

Let (M, J) be an almost complex manifold of dimension $2m \geq 6$. To study local pseudoholomorphic submanifolds of dimension 4 we consider

$$\mathcal{E} = \{(x, y, \Phi) \in J^1(\mathbb{R}^4, M) \mid x \in \mathbb{R}^4, y \in M, \Phi \in T_x^*\mathbb{R}^4 \otimes T_yM, \\ J\,\mathrm{Im}\,\Phi = \mathrm{Im}\,\Phi\}.$$

Although it is not necessary we will restrict to the regular part of the above equation specified by the condition $\mathrm{rk}(\Phi) = 4$ (the irregular part corresponds to singularities). We use the same letter $\mathcal{E}$ for this smaller equation. The symbol of the equation at a point $x_1 = (x, y, \Phi)$ is

$$g_1(x_1) = \{\phi \in T_x^*\mathbb{R}^4 \otimes T_yM \mid J\phi - \phi\tilde{J} \in \mathrm{Im}\,\Phi\},$$

where $\tilde{J}$ is an almost complex structure on $T_x\mathbb{R}^4$ depending on x_1. In what follows, we write $\tau = T_x\mathbb{R}^4$, $\zeta = T_yM$, $\Pi = \mathrm{Im}\,\Phi$. We have:

$$g_k(x_1) = \{\phi \in S^k\tau^* \otimes \zeta \mid J\phi(\xi_1, \dots, \xi_i, \dots, \xi_k) - \phi(\xi_1, \dots, \tilde{J}\xi_i, \dots, \xi_k) \in \Pi\ \forall i\},$$

where $\tilde{J}$ is the same as for x_1. Denoting $g_k^\Pi = S^k\tau^* \otimes \Pi \subset g_k$, we get $g_k^\nu := g_k/g_k^\Pi = S^k\tau^* \otimes_{\mathbb{C}} \nu$ – the set of $\tilde{J}$-J linear maps $S^k\tau \to \nu$, where $\nu = \zeta/\Pi$. In particular, $g_k \neq \{0\}$ and $\mathcal{E}$ is an equation of infinite type.

The symbolic system $\{g_k\}$ is involutive: $H^{i,j} = 0$ for $i > 0$. In fact both the systems g_k^Π and g_k^ν are involutive by the Poincaré lemma: one for the de Rham and the other for the Dolbeault differentials. The only non-zero second cohomology group occurs at the last term of the Spencer sequence

$$0 \to g_2 \xrightarrow{\delta_2} g_1 \otimes \tau^* \xrightarrow{\delta_1} \Lambda^2\tau^* \otimes \zeta \to 0. \tag{1}$$

The space $H^{0,2}(\mathcal{E}, x_1)$ consists of all $\tilde{J}$-J anti-linear (2, 1)-tensors modulo Π. In fact we can identify $g_2 = \mathrm{Ker}\,\delta_1$ with the preimage of $S^2\tau^* \otimes_{\mathbb{C}} \nu$ under the projection $S^2\tau^* \otimes \zeta \to S^2\tau^* \otimes \nu$. Then the cokernel of δ_1 in sequence (1) is the space $H^{0,2}(\mathcal{E}) \simeq \Lambda^2\tau^* \otimes_{\bar{\mathbb{C}}} \nu$ of dimension $2(m-2)$.

Proposition 11. *The Weyl tensor W_1 at a point $x_1 = (x, y, \Phi)$ with* $\operatorname{Im}\Phi = \Pi$ *is given by*

$$\xi_1 \wedge \xi_2 \mapsto \frac{1}{4} N_J(\Phi\xi_1, \Phi\xi_2) \bmod \Pi, \quad \xi_1, \xi_2 \in \tau.$$

Proof. Let $[\xi] \in \nu$ be the projection $\xi \bmod \Pi$. The projection $\Lambda^2\tau^* \otimes \zeta \to \operatorname{CoKer}\delta_1 = H^{0,2}$ associates to every skew-symmetric (2,1)-tensor P its anti-linear $(-,-)$ part $\bmod \Pi$. Since $\Phi : \tau \xrightarrow{\sim} \Pi$ is an isomorphism, we can identify $P \in \Lambda^2\Pi^* \otimes \zeta$. Then the $(-,-)$ part of $[P] \in \Lambda^2\Pi^* \otimes \nu$ is:

$$[P^{--}](\xi, \eta) = \big[(P(\xi, \eta) - P(J\xi, J\eta) + JP(J\xi, \eta) + JP(\xi, J\eta))/4\big], \quad \xi, \eta \in \Pi.$$

The metasymplectic structure Ω_1 equals the curvature of the Cartan distribution $\mathcal{C}_1 = \operatorname{Ker}(dy - \Phi dx)$. Thus the restriction $\Omega_1|_H$ is equal to the torsion T_∇ of a linear connection ∇ on M determining the horizontal distribution H: In fact the curvature of an affine connection is the sum of the curvature and the torsion of the corresponding linear connection [KN], but in the calculation of Ω_1 [Ly] the curvature part is cancelled.

The distribution H is a Cartan connection (tangent to the equation, $H \subset \mathcal{C}_1$) iff it is generated by a linear connection ∇ preserving the almost complex structure: $\nabla J = 0$. But the $(--)$ part of any almost complex connection is canonical $T_\nabla^{--} = \frac{1}{4}N_J$ ([Li]). Since the Weyl tensor $W_1 = [\Omega_1|_H]$ is the $[(--)]$ part of T_∇, we are done. □

Let us give another indication of the fact that the equality $W_1 = 0$ is equivalent to $N_J(\Pi, \Pi) \subset \Pi$. It is obvious that whenever we have a 2-jet of a pseudoholomorphic mapping of $(\mathbb{R}^4, x, \tilde{J})$ into (M, y, J), the 1-jet of it preserves the Nijenhuis tensor: $\varphi_* \circ N_{\tilde{J}} = N_J \circ \Lambda^2\varphi_*$.

On the other hand suppose we have 1-jet of a map, i.e. a linear map $\Phi : T_x\mathbb{R}^4 \to T_yM$ with image $\Pi = \operatorname{Im}\Phi$ that is J, N_J-invariant. Induce the complex structure and Nijenhuis tensor on $T_x\mathbb{R}^4$ by Φ. Extend the complex structure to an almost complex structure $\tilde{J}$ in a neighborhood of x so that the Nijenhuis tensor $N_{\tilde{J}}$ at x coincides with the prescribed N_J. Then by Theorem 1 of [Kr1] the map can be changed so that its 2-jet maps $(\mathbb{R}^4, x, \tilde{J})$ into (M, y, J).

2.3. First prolongation-projection

Not all points of $\mathcal{E}$ have prolongations. Those that do form a new equation $\tilde{\mathcal{E}} = \pi_{2,1}(\mathcal{E}^{(1)})$. Due to the above calculations it is described as follows:

$$\tilde{\mathcal{E}} = \{(x, y, \Phi) \mid \Pi = \operatorname{Im}\Phi \text{ satisfies } J\Pi = \Pi, N_J(\Pi, \Pi) \subset \Pi\}.$$

In other words, the fiber $\tilde{\mathcal{E}}_{x,y}$ consists of all possible parametrizations Φ of the Grassmannian $\operatorname{Gr}_4(T_yM, J, N_J)$.

By Proposition 9 the dimension of $\tilde{\mathcal{E}}_{x,y}$ is $4^2 + 2(m-2)$ for generic J and (x, y), which certainly coincides with $\dim g_1 - \dim H^{0,2}$. The symbol at a point $x_1 = (x, y, \Phi)$ can be described as follows:

$$\tilde{g}_1(x_1) = \{\phi \in \tau^* \otimes \zeta \mid [J\phi] = [\phi\tilde{J}],\ [N_J(\phi, \Phi)] + [N_J(\Phi, \phi)] = [\phi N_{\tilde{J}}]\}, \tag{2}$$

where $\tilde{J}$, $N_{\tilde{J}}$ are induced by Φ and $[\cdot]$ denotes the class mod Π. Since N_J preserves Π, there is an induced map $N_J^\nu : \Pi \times \nu \to \nu$. Let us fix a totally real subbundle of (ν, J), defining a conjugation. Then $\bar{N}_J^\nu \in \mathrm{Hom}_{\mathbb{C}}(\Pi, \mathrm{Aut}_{\mathbb{C}}(\nu))$. So introducing $\varphi = [\phi \circ \Phi^{-1}] \in \mathrm{Hom}_{\mathbb{C}}(\Pi, \nu)$ we rewrite the second condition of (2) as

$$\bar{N}_J^\nu \wedge \varphi = \bar{\varphi}(N_J).$$

Note that $\mathrm{Tr}^{\mathbb{C}}(\bar{N}_J^\nu)$ is a well-defined $\mathbb{C}$-valued 1-form on Π. In general position it is non-zero on $\Pi_0 = N_J(\Pi, \Pi)$ and so this (complex) line is transversal to the line $\Pi_1 = \mathrm{Ker}\,\mathrm{Tr}^{\mathbb{C}}(\bar{N}_J^\nu)$. Thus $\Pi_0 \oplus \Pi_1 = \Pi$. The restricted Nijenhuis tensor $N_J|_\Pi \in \Lambda^2\Pi^* \otimes \Pi$ is fixed by a complex basis $X, Y \in \Pi$, subject to condition $N_J(X, Y) = X \in \Pi_0$, $Y \in \Pi_1$ (§1.4). This basis is defined up to a change $X \mapsto \lambda e^{it} X$, $Y \mapsto e^{-2it} Y$. We fix non-uniqueness by the requirement $\mathrm{Tr}^{\mathbb{C}}\,\bar{N}_J^\nu(X) = 1$. Using this basis we rewrite the above condition:

$$\bar{\varphi}(X) = \bar{N}_J^\nu(X)\varphi(Y) - \bar{N}_J^\nu(Y)\varphi(X). \tag{3}$$

Let us add non-degeneracy of $\bar{N}_J^\nu(X) \in \mathrm{Aut}_{\mathbb{C}}(\nu)$ to the genericity assumptions for the structure J.

The equation $\tilde{\mathcal{E}}$ is of infinite type: $\tilde{g}_k \neq \{0\}\,\forall k$. In fact $\tilde{g}_k \supset \tilde{g}_k^\Pi = S^k\tau^* \otimes \Pi$. Moreover, the quotient $\tilde{g}_k^\nu = \tilde{g}_k/\tilde{g}_k^\Pi$ has dimension $2(m-2)$ for $k = 1$ and is zero for $k \geq 2$. In fact for $k = 1$ this follows directly from (3) and if $\varphi_2 \in S^2\Pi^* \otimes_{\mathbb{C}} \nu$ is an element of $\tilde{g}_2^\nu$, then for any $\xi \in \Pi$ we have:

$$\begin{aligned} \varphi_2(\xi, X) &= N_J^\nu(X)\varphi_2(\xi, Y) - N_J^\nu(Y)\varphi_2(\xi, X) \\ &= JN_J^\nu(X)\varphi_2(J\xi, Y) - JN_J^\nu(Y)\varphi_2(J\xi, X) \\ &= J\varphi_2(J\xi, X) = -\varphi_2(\xi, X), \end{aligned}$$

implying $\varphi_2 \equiv 0$. Thus $\tilde{g}_k = (\tilde{g}_2)^{(k-2)} = (\tilde{g}_2^\Pi)^{(k-2)} = \tilde{g}_k^\Pi\ \forall k \geq 2$ and $\tilde{g}_k^\nu = 0$.

The system $\tilde{g}_k \subset S^k\tau^* \otimes \zeta$ has the same cohomology as $\tilde{g}_k^\nu \subset S^k\Pi^* \otimes \nu$ and $\dim H^{0,2}(\tilde{\mathcal{E}}) = 4(m-2)$. Thus we get $2(m-2)$ new conditions, which single out the prolongable jets from $\tilde{\mathcal{E}}$. These new conditions are elements of the group $H_\nu^{0,2} := H^{0,2}(\tilde{\mathcal{E}})/H^{0,2}(\mathcal{E})$, where $H^{0,2} \subset \tilde{H}^{0,2}$ due to the inclusion $\tilde{g}_1^\nu \subset g_1$. This group can be identified with $\Lambda^2_{\mathbb{C}}\Pi^* \otimes_{\mathbb{C}} \nu$ (and also with $\Lambda^2\Pi_0^* \otimes \nu$). Denote by $\Xi : \Lambda^2\Pi^* \otimes \nu \to H_\nu^{0,2}$ the projection along $(\Lambda^2_{\mathbb{C}}\Pi^* \otimes_{\bar{\mathbb{C}}} \nu) \oplus \delta(\Pi^* \otimes \tilde{g}_1)$.

Let's call $\Pi_x \in \mathrm{Gr}_4(M, J, N_J)$ regular if the Grassmannian is a smooth manifold at Π_x and the projection to M is non-degenerate. By a semi-connection on $\pi : \mathrm{Gr}_4(M, J, N_J) \to M$ we understand a distribution $H \subset T\,\mathrm{Gr}_4(M, J, N_J)$ with $\pi_* : H_\Pi \xrightarrow{\sim} \Pi$. Its curvature is a 2-form $\Theta_H \in \Lambda^2 H^* \otimes (T\,\mathrm{Gr}_4(M, J, N_J)/H)$ defined by $\Theta_H(\xi, \eta) = [\xi, \eta] \mod H$. So at a regular point Π of the Grassmannian we obtain the tensor $\pi_*\Theta_H(\Pi) \in \Lambda^2\Pi^* \otimes \nu$.

The following result is obtained by a calculation as in Proposition 11:

Proposition 12. *The Weyl tensor $\tilde{W}_1$ of PDE $\tilde{\mathcal{E}}$ for a generic structure J at a regular generic point $x_1 = (x, y, \Phi)$ is given by the formula:*

$$\xi_1 \wedge \xi_2 \mapsto \Xi(\pi_* \Theta_H(\Pi))(\Phi\xi, \Phi\eta), \quad \xi_1, \xi_2 \in \tau. \qquad \square$$

2.4. Second prolongation-projection

The points of $\tilde{\mathcal{E}}$ having prolongations determine the next equation $\pi_{2,1}(\tilde{\mathcal{E}}^{(1)})$, which provided the almost complex structure is generic is given by

$$\hat{\mathcal{E}} = \{(x, y, \Phi) \mid \Pi = \operatorname{Im} \Phi \in \operatorname{Gr}_4(T_x M, J, N_J),\ \Xi(\pi_* \Theta_H(\Pi)) = 0\}.$$

The symbol of this equation is $\hat{g}_1 = \tau^* \otimes \Pi$ and its prolongations are $\hat{g}_k = S^k \tau^* \otimes \Pi$. Thus the fiber $\hat{\mathcal{E}}_{x,y}$ modulo the reparametrizations group is discrete ($\hat{g}_1^\nu = 0$) and is locally presented at regular points by a finite number of distributions – sections of $\operatorname{Gr}_4(TM, J, N_J)$. The following statement is now obvious:

Proposition 13. *The Spencer group $H^{0,2}(\hat{\mathcal{E}}) = \Lambda^2 \Pi^* \otimes \nu$. The Weyl tensor $W_1(\hat{\mathcal{E}}; x_1)$ is the curvature of the corresponding distribution through $\Pi = \operatorname{Im} \Phi \in \operatorname{Gr}_4(T_x M, J, N_J)$ determined by the point x_1.* $\square$

Thus for a generic almost complex structure J the equation $\hat{\mathcal{E}}$ has prolongations only at finite number of points (where $\hat{W}_1 = 0$) among the open dense subset of regular points. Thus there pass no local PH-submanifolds L^{2n} through any regular point.

Remark. The tensor invariants algebra of an almost complex structure J does not simplify the proof. Due to [Kr1] it begins with $\mathcal{A}_J^\infty = \langle J, N_J, N_J^{(2)}, \dots \rangle$, $N_J^{(2)} \in \Lambda^2(\Lambda^2 T_x^* M) \otimes T_x M$. For every invariant λ we associate its quotient $[\lambda] \in \otimes^k \Pi^* \otimes \nu$, $\nu = TM/\Pi$. Thus the zeros of $[J], [N_J]$ is the N_J-Grassmannian, but if $N_J(\Pi, \Pi) \subset \Pi$ and $\dim \Pi = 4$, then $[N_J^{(2)}] \in \Lambda_{\mathbb{C}}^2(\Lambda_{\mathbb{C}}^2 \Pi^*) \otimes \nu$ that is zero.

2.5. Non-existence of submanifolds

So far we have been considering only regular points. The set of non-regular points form a stratified submanifold $\Sigma \subset M$ of positive codimension for a generic almost complex structure J. This submanifold carries a non-holonomic almost complex structure ($\mathcal{D} = T\Sigma \cap JT\Sigma$, $J|_{\mathcal{D}}$) (defined only separately for each stratum), which is generic for a C^r-generic structure J. Note that adding a standard integrability condition to J we get the well-known CR-structure (non-holonomic complex structure).

Now the non-existence of higher-dimensional PH-submanifolds follows from any of the following non-existence statements:

– *The only integral submanifolds of a generic distribution $\mathcal{D}$ on Σ are curves.*
– *A generic non-holonomic almost complex structure contains no PH-curves (i.e. surfaces tangent to $\mathcal{D}$ and J-invariant).*

– *A generic non-holonomic almost complex structure contains no PH-submanifolds of dimension* $2n \in [4, \mathrm{rk}(\mathcal{D})]$.

The proofs are obtained by the above approach. We will indicate the proof of the simplest last statement, since it requires the weakest C^1-topology, independently for each stratum.

Consider a stratum of dimension s, which we denote by Σ^s. Let the distribution $\mathcal{D} \subset T\Sigma^s$ have rank $2t < s$. Define the Levi form $L_{\mathcal{D},J} \in S^2\mathcal{D}^* \otimes T\Sigma^s/\mathcal{D}$ of a non-holonomic almost complex structure $(\Sigma^s, \mathcal{D}^{2t}, J)$ by $L_{\mathcal{D},J}(\xi, \eta) = \Theta_{\mathcal{D}}(\xi, J\eta) + \Theta_{\mathcal{D}}(\eta, J\xi)$, where $\Theta_{\mathcal{D}} \in \Lambda^2\mathcal{D}^* \otimes T\Sigma^s/\mathcal{D}$ is the curvature of the distribution $\mathcal{D}$.

At a generic point the vector valued quadric $L_{\mathcal{D},J}$ is non-degenerate (the structure is pseudoconvex), whence no PH-curves passes through it. Searching for $2n$-dimensional PH-submanifolds, we require that $L_{\mathcal{D},J}$ has a $2n$-dimensional null-subspace $V \subset \mathcal{D}$. Generically this specifies a stratified submanifold $\Lambda^r \subset \Sigma^s$ of codimension $n(2n+1)(s-2t)$ and dimension $r = s - n(2n+1)(s-2t)$. There is no more than q-dimensional family of $2n$-dimensional $L_{\mathcal{D},J}$-null subspaces at the points of the stratum $\Lambda^r_q \subset \Lambda^r$ of dimension $r - q$. For each such subspace V the J-invariance condition $JV = V$ gives $2n(t-n)$ additional equations.

One easily checks that $2n(t-n) > r - q + q = r$ for $s > 2t > 2n > 3$. This implies non-existence of $2n$-dimensional local PH-submanifolds in Λ^r_q for all q, r, s in generic situation and thus finishes the proof of the embedding part of the main theorem.

2.6. Non-existence of automorphisms

To study the group of PH-automorphisms $\mathrm{Aut}(M, J)$ one considers PDE $\mathcal{E} = \{(x, y, \Phi) \mid \Phi \in T^*_xM \otimes T_yM, \Phi J = J\Phi, \mathrm{rk}\,\Phi = 2m\}$. Its 1st prolongation-projection $\tilde{\mathcal{E}} = \pi_{2,1}\mathcal{E}^{(1)}$ consists of maps preserving both structures J and N_J.

In the case $m \geq 4$ with C^1-generic J this equation has only one solution id_M because the dimension of the orbit space of the Nijenhuis tensors $[\mathcal{N}(V, J) = \Lambda^2_{\mathbb{C}} V^* \otimes_{\bar{\mathbb{C}}} V]/\,\mathrm{Gl}_{\mathbb{C}}(V)$ is greater than $\dim V = 2m$.

In the case $m = 3$ as follows from Theorem 7 the dimension of the orbit space is 2. So the 6-dimensional space is fibered by 4-dimensional varieties Σ_α on each of which the type of the Nijenhuis tensor is fixed. For C^1-generic structure J there exists an open dense subset $\Sigma' \subset \Sigma$ with the following properties. On Σ' we have two 2-dimensional and transversal distributions $Z^2_\alpha = T\Sigma_\alpha \cap JT\Sigma_\alpha$ and $Y^2_\alpha = T\Sigma_\alpha \cap \Phi_1(Z^2)$ (with Φ_1 from §1.4). The first of these distributions is J-invariant and the second is not. Moreover $X^2_\alpha := N_J(Y^2_\alpha, Z^2_\alpha) \subset Y^2_\alpha + JY^2_\alpha$ has zero intersection with both Y^2_α and JY^2_α.

Let us define the map $\psi : Y^2_\alpha \to Y^2_\alpha$, $\eta \mapsto (\eta + J\eta' \in X^2_\alpha) \mapsto \eta'$. Consider for simplicity only the case when its spectrum is real and simple. Then we have two eigenvectors $\eta_1, \eta_2 \in Y^2_\alpha$. There are canonical vectors $\zeta_1, \zeta_2 \in Z^2_\alpha$ such that $\mathrm{pr}_Y \circ N_J(\zeta_i, \eta_i) = \eta_i$, where pr_Y is the projection of $X^2_\alpha \subset Y^2_\alpha \oplus JY^2_\alpha$ to the first component. For a C^2-generic J the obtained frame on Σ' (defined up to rescaling of η_i and permutation of the indices 1, 2) is rigid, i.e. it admits no automorphisms save for id_M.

In the case $m = 2$ and C^2-generic almost complex structure J a canonical $\{e\}$-structure (absolute parallelism) on M^4 was constructed in [Kr2]. Recall briefly the construction (yielding a solution to the classification problem). We define two distributions on M^4: $\Pi^2 = \operatorname{Im} N_J$ and its derivative $\Pi^3 = [\Pi^2, \Pi^2]$. We construct the frame $\xi_1 \in \Pi^2$, $\xi_2 = J\xi_1 \in \Pi^2$, $\xi_3 = [\xi_1, \xi_2] \in \Pi^3/\Pi^2$ and $\xi_4 = J\xi_3 \in TM/\Pi^3$ by the requirement $N_J(\xi_1, \xi_3) = \xi_1$. The pair (ξ_1, ξ_2) is defined up to a sign and the pair (ξ_3, ξ_4) is canonical. For a C^3-generic structure J the frame $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ is rigid.

Thus we proved the second part of the main theorem.

2.7. Non-existence of submersions

Let us call a linear Nijenhuis tensor N on V *projectible* if $N(V, W) \subset W$ for some proper $W \subset V$. Because $\pi_* N_J(\xi, \eta) = N_J(\pi_*\xi, \pi_*\eta)$ we have (cf. [Kr2]):

Lemma 14. *Let $\pi : (M^{2m}, J) \to (L^{2n}, \tilde{J})$ be a PH-submersion and $0 < n < m$. Then the Nijenhuis tensor N_J on T_xM is projectible with $W_x = \operatorname{Ker} \pi_*(x)$.* □

A generic Nijenhuis tensor is not projectible for $m > 2$. In the case $m \geq 4$ this follows from dimensional reasons and for $m = 3$ from non-degeneracy of generic N (§1.4). Consider the case $m = 2$. The only non-zero Nijenhuis tensor N is projectible and $W^2 = N(V, V)$. But for a C^2-generic structure J the canonical distribution $\Pi^2 = \operatorname{Im}(N_J)$ on M^4 is non-integrable contrary to $W^2 = \operatorname{Ker} \pi_*$. This finishes the proof of the main theorem.

Note that for $m \geq n + 2$ the third statement of the theorem follows from the first, but the topology becomes finer.

3. Other results

3.1. Comparison Theorems

$2n$-dimensional PH-submanifolds are absent for generic J, $1 < n < m$. On the contrary, abundance of such submanifolds implies the integrability of the structure J. A similar thing happens to the automorphisms and submersions:

Theorem 15. *(M, J) is a complex manifold iff it satisfies one of the conditions:*

1. $\mathrm{Gr}_{2n}(M, J, N_J) = \mathrm{Gr}_n^{\mathbb{C}}(M, J)$*, i.e. in every complex direction a 1-jet of* $2n$*-dimensional PH-submanifold passes,* $1 < n < m$.
2. *The stabilizer of* $\mathrm{Aut}_{loc}(M, J)$ *has dimension* $2m^2$ *at each* $x \in M$ *and therefore is equal to* $\mathrm{Gl}_{\mathbb{C}}(T_xM)$.
3. *The set of 1-jets of PH-submersions with the kernels of* $\dim_{\mathbb{C}} = n$ *equals* $\mathrm{Gr}_n^{\mathbb{C}}(M, J)$.

Proof. Condition 1 means the canonical section of Proposition 2 vanishes. It is enough to require vanishing of Γ_N on an open subset, since then the coordinate expression shows $N_J = 0$ and thus the structure J is integrable.

For any $\Pi^4 \in \mathrm{Gr}_4(T_xM, J, N_J)$ the orbit of the isotropy subgroup $St_x \cdot \Pi^4$ is open in $\mathrm{Gr}_2^{\mathbb{C}}(T_xM)$ and the arguments of part 1) apply. The complexification of $\mathrm{Gr}_4(T_xM, J, N_J)$ is always non-empty and statement 2) follows. The third statement follows from the first. □

Part (2) of the above theorem is similar to the following combination of statements from [KO]:

Theorem 16. *Let $D^{2m} \Subset (D', J)$ be a compactly embedded almost complex ball and (D', J) be tamed by a symplectic structure. Then* $\dim \mathrm{Aut}(D, J) \le 2m + m^2$ *and the equality holds iff the almost complex structure J is integrable and (D, J) is biholomorphic to (B^{2m}, J_0).* □

Remark. A similar result to the 1st statement of Theorem 15 is proved in [M]. We formulate it in the language of PDEs as in §2.1: Let $\mathcal{E} \subset J^1(L, M)$ be an equation with the symbols as for the Cauchy-Riemann equation (for the complex maps between L and M). Suppose the first prolongation exists for all points of $\mathcal{E}$ and is regular. In addition suppose that $n = \frac{1}{2}\dim L > 1$ and if $m = \frac{1}{2}\dim M = 1$ we impose an additional topological condition. Then the equation is equivalent to the Cauchy-Riemann equation by a transformation induced from $J^0(L, M)$ (i.e. by a change of independent and dependent variables).

The idea is as follows (the approach of [M] is different): At first we notice that since the Cauchy-Riemann symbolic system is involutive and the only obstruction to prolongation vanishes, the system is formally integrable. In fact $\mathcal{E}$ is smoothly integrable because it is elliptic (another way is to introduce an almost complex structure on the equation and to use the integrability criterion). This gives an equivalence of equations, that should boil down to $J^0(L, M)$ in the case $n, m > 1$: Because the equations are normal (in the terminology of [KLV]), they are rigid. For $m = 1$ there are $\mathbb{C}$-contact Lie transformations and to exclude them one introduces a topological condition.

The paper [M] is based on a completely different approach than ours (it uses the theory of exterior differential systems). The equation on PH-submanifolds, which we considered, is quasilinear contrary to a more general setting of McKay. The quasi-linearity is obtained as a corollary of the hypotheses.

Then the equation $\mathcal{E}$ can be seen as an equation for PH-submanifolds and our geometric approach works as the following formal trick shows (invented by Gromov for PH-curves [Gr]): Let $\bar\partial_{J_L,J_M} f = \frac{1}{2}(df + J_M(f) \circ df \circ J_L)$ be the non-linear Cauchy-Riemann operator corresponding to almost complex structures J_L on L and J_M on M, $f \in C^\infty(L, M)$. Consider a section $g \in C^\infty(T^*L \otimes_{\bar{\mathbb{C}}} TM)$. Then the non-homogeneous equation $\bar\partial_{J_L,J_M} f = g$ is equivalent to the fact that the map $\mathrm{id_L} \times f : (L, J_L) \to (L \times M, J_g)$ is pseudoholomorphic, where the new almost complex structure is defined by $J_g(\xi, \eta) = (J_L\xi, J_M\eta + 2g\xi)$.

3.2. *Remark on a result of Donaldson*

We can view locally an almost complex structure on M as a fiberwise deformation $J \in \mathrm{End}(TM)$ of a complex structure i. For two such structures J_L and J_M a PH-morphism φ between them, i.e. a map whose differential φ_* commutes with J-multiplication, can, as we have proved, cease to exist. However the quantization theories predict the existence of a deformed commutativity:

$$\begin{array}{ccccc} TL & \xrightarrow{Q_L} & TL & \xrightarrow{\varphi_*} & TM \\ \downarrow{\scriptstyle J_L} & & & & \downarrow{\scriptstyle J_M} \\ TL & \xrightarrow{\varphi_*} & TM & \xrightarrow{Q_M} & TM \end{array} \tag{4}$$

for some bundle morphism Q_L over $\mathbf{1}_L$ close to $\mathbf{1}_{TL}$ and similarly for Q_M.

Theorem 17. *Let φ_0 be a PH-map that is an embedding or a submersion. For every map φ C^1-close to φ_0 there exist two morphisms $Q_L = Q_L(\varphi)$ over id_L and $Q_M = Q_M(\varphi)$ over id_M, each close to the identity morphism of the corresponding tangent bundle, such that diagram (4) commutes.*

Proof. Since we deal with bundle morphisms the Theorem follows from the following linear algebra statement:

Lemma 18. *For every linear map $\Phi : V \to W$ close to a complex linear map $\Phi_0 : V \to W$ there exist automorphisms $Q_V : V \to V$, $Q_W : W \to W$ such that $Q_W \Phi J_V = J_W \Phi Q_V$. One can achieve $\mathrm{rk}\, Q_V = \mathrm{rk}\, Q_W = \mathrm{rk}\, \Phi_0$. If in addition Φ_0 is into or onto, the maps Q_V, Q_W can be chosen close to the identities.*

Fix J-invariant splittings $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$ such that $\Phi_0(X_1, X_2) = (X_1, 0)$, $\dim V_1 = \dim W_1 = \mathrm{rk}\, \Phi_0$. Let $W_1' = \Phi(V_1)$, $W_1'' = JW_1'$. Denote by $\tilde{\Phi}$ the composition $\mathrm{proj} \circ \Phi : V \to W = W_1' \oplus W_2 \to W_1'$. Let $V_2' = \mathrm{Ker}\, \tilde{\Phi}$, $V_2'' = JV_2'$. By our assumption V_2', V_2'' are close to V_2 and W_1', W_1'' are close to W_1. So the restriction of $\tilde{\Phi}$ is an isomorphism $\hat{\Phi} : V_1 \to W_1'$. Now we have the commutative diagram (Φ does not preserve grading):

$$\begin{array}{ccccc} V_1 \oplus V_2'' & \xrightarrow{Q_V} & V_1 \oplus V_2' & \xrightarrow{\Phi} & W_1' \oplus W_2 \\ \downarrow{\scriptstyle J_V} & & & & \downarrow{\scriptstyle J_W} \\ V_1 \oplus V_2' & \xrightarrow{\Phi} & W_1' \oplus W_2 & \xrightarrow{Q_W} & W_2'' \oplus W_2. \end{array}$$

In the general case we define $Q_V = q_V \oplus 0$, $Q_W = P_W \oplus 0$. Here $P_W : W_1' \to W_1''$ is the projection along W_2 and $q_V \in \mathrm{Gl}(V_1)$ is given by $-q_V(\xi) = \hat{\Phi}^{-1} J_W P_W \hat{\Phi} J_V(\xi)$. The commutativity follows.

Whenever Φ_0 is into, we have $V_2 = 0$, $Q_V = q_V$ and the commutativity is achieved with $Q_W = P_W \oplus \mathbf{1}$. If Φ_0 is onto, $W_2 = 0$, $Q_W = \mathbf{1}$ and we let $Q_V = q_V \oplus P_V$, where $P_V : V_2'' \to V_2'$ is the projection along V_1. □

Now we define a map Φ to be approximately pseudoholomorphic if it satisfies the diagram (4) with close to the identities Q: $\rho_L(Q_L)$, $\rho_M(Q_M) < \hbar$, where $\rho_U = d(\mathbf{1}_{TU}, \cdot)$ are distances to the identity morphisms.

Such maps perfectly exist as was shown by Donaldson in [D] for embeddings (our definition of approximately pseudoholomorphic is equivalent to his). Moreover if the almost complex structure J_M is compatible with a symplectic structure on M then an approximately PH-submanifold $\varphi(L)$ is necessarily symplectic. The derivatives of Q_M provide a (2, 1)-tensor related to the Nijenhuis tensor N_{J_M} and this suggests investigation of approximate N_J-Grassmannian whose solutions are symplectic submanifolds for any compatible symplectic structure.

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