Mayer brackets and solvability of PDEs – I

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Abstract

A solvability of a system of partial differential equations with two variables is investigated. The smooth solvability conditions are formulated in terms of (generalized) Mayer brackets. [1]

Introduction

Various brackets were investigated in connection with differential equations since the time of Lagrange. Monge and Lie's geometrization of first order partial differential equations gave new intrinsic meaning of the brackets, which were further studied by A. Mayer, F. Klein and others.

All these brackets have general nature. They do not depend on a particular differential equation but only on a special geometric structure common for all differential equations of the given type. Say the Jacobi and Lagrange brackets depend on symplectic and contact structures respectively, which form the geometrical base for 1-st order differential equations. There are some approaches to get higher-jets analogs in the geometric theory of PDEs ([KS]).

The brackets serve as a machinery for integration of differential equations. The more instructive and geometrical way to observe obstructions for integrability produces the Weyl tensor ([Ly]) with values in the second Spencer cohomology group. This tensor is also the first obstruction for equivalence of differential equations ([KL]).

In this paper we introduce a special bracket $[,]$, which we call *Mayer bracket* (and which is actually a proper generalization of the classical Mayer bracket – see the historical remark at the end of the paper). This higher Mayer bracket depends actually on the coordinate representation but its restriction to the corresponding equation is invariant. In fact the restriction coincides with the restriction of the higher Jacobi bracket.

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*The first author is supported by grants NWO-RFBR 047-008-005, RFBR 99010147.

[1]MSC numbers: 35N10, 58A20, 58H10; 35A30. Keywords: Mayer bracket, Spencer cohomology, Weyl tensor, characteristics, symbols, solvability of PDEs.

For PDEs on two-dimensional manifolds we express the Weyl tensor via the Mayer bracket. In this case the dimension of the corresponding Spencer group is one and so the bracket can be considered as scalar-valued. Our main result is

Theorem. *Consider a system $\mathcal{E}$ of partial differential equations*

$$\begin{cases} F\left(x,y,u(x,y),\dots,\frac{\partial^k u}{\partial x^k},\frac{\partial^k u}{\partial x^{k-1}\partial y},\dots,\frac{\partial^k u}{\partial y^k}\right) = 0 \\ G\left(x,y,u(x,y),\dots,\frac{\partial^l u}{\partial x^l},\frac{\partial^l u}{\partial x^{l-1}\partial y},\dots,\frac{\partial^l u}{\partial y^l}\right) = 0 \end{cases}$$

If the equations $F=0$ and $G=0$ have no common characteristics and at least one of them has distinct characteristics then the system is smoothly integrable if and only if the Mayer bracket vanishes due to the system:

$$[F,G]_{\mathcal{E}} = 0.$$

Remark 1. *It seems that the condition of distinct characteristics is superfluous. At least this condition is not required in the case when one of the equations has order one and when both F,G are of the second order.*

This theorem can be used in solving PDEs by means of overdetermination (for 1-st order equations it is the classical Lagrange-Charpit method): To solve some differential equation on two-dimensional manifold one adds a new equation, which is in involution with the initial equation with respect to Jacobi or Mayer bracket, and then finds solutions of the resulting system by the Frobenius theorem applied to the Cartan distribution.

For example, the new equation can be chosen from the higher symmetries of the initial equation. Actually one easily checks that $[F,G]=X_F^{(k)}G$ on the equation $\mathcal{E}=\{\mathcal{D}_\sigma F=0, G=0\}_{|\sigma|<k}$, where $F\in C^\infty(J^1M), G\in C^\infty(J^kM)$ and $X_F^{(k)}$ is the prolongation of contact vector field X_F.

Remark 2. *In fact the methods developed in proof of the main theorem have a wider range of applications. They produce integrability criteria for arbitrary overdetermined PDEs $\mathcal{E}\subset J^k(\pi)$ on a bundle π over manifold M^n of dimension $n\geq 2$ provided the characteristic variety $P\,\mathrm{Char}(\mathcal{E})\subset P(^{\mathbb{C}}T^*M)$ has discrete "fiber" $P\,\mathrm{Char}_{x_k}(\mathcal{E})$ over any $x_k\in\mathcal{E}$.*

The paper is organized as follows. In first sections we give a brief introduction to the geometric theory of PDEs and Spencer cohomologies ([KLV], [S]) in a suitable for us form. Then we define Weyl tensor ([Ly]) and outline methods for its calculation. We introduce Mayer brackets in coordinates, give an invariant point of view and finally prove the main theorem. Note that we consider only the case of jets of the trivial one-dimensional bundle over the manifold M, which we treat in details.

As an application of our method we give a simple proof of the linearization construction for parallelizable 3-webs on 2-dimensional manifolds, which was presented recently by J. Grifone, Z. Muzsnay and J. Saab.

1. Obstructions for integrability in geometric theory of PDEs

1.1. Prolongations

Let M be a smooth manifold. Two functions $f_1, f_2 \in C^\infty(M)$ are said to have the same jet at $x \in M$ if they differ by a quantity of order $k+1$: $f_1 - f_2 \in \mu_x^{k+1}$, where $\mu_x = \{f \in C^\infty(M) \mid f(x) = 0\}$ is the maximal ideal of the point x. The equivalence class is called the k-jet $x_k = [f]_x^k$.

Denote the space of all k-jets of functions on M by $J^k M = \bigcup_{x\in M} J^k_x M$. There are natural projections $\pi_{k,l} : J^k M \to J^l M$ and $\pi_k : J^k M \to M$. Any smooth function $f \in C^\infty(M)$ induces a section $j_k f : x \mapsto [f]_x^k$ of the bundle π_k.

The fiber $F(x_{k-1}) = \pi_{k,k-1}^{-1}(x_{k-1})$ has a natural affine structure associated to the vector space $S^k \tau_x^*$, where $\tau_x = T_x M$. We write $F(x_{k-1}) \simeq S^k \tau_x^*$.

A *differential equation* of order k of one unknown function can be considered as a submanifold $\mathcal{E} \subset J^k M$. It is usually assumed that $\pi_{k,k-1} : \mathcal{E} \to J^{k-1} M$ is a bundle projection (in particular surjective). We define the r^{th} prolongation of the equation $\mathcal{E}$ by the formula

$$\mathcal{E}^{(r)} = \{x_{k+r} = [f]_x^{k+r} \in J^{k+r} M | j_k f(M) \text{ is tangent to } \mathcal{E} \text{ at } x_k \text{ with order} \geq r\}.$$

PDE $\mathcal{E} \subset J^k M$ is called *formally integrable* if $\pi_{k+s,k+s-1} : \mathcal{E}^{(s)} \to \mathcal{E}^{(s-1)}$ are smooth bundle projections. Let us also consider the map $\emptyset \to N$ as bundle projection with empty fibers. Then we should additionally require $\mathcal{E}^{(s)} \neq \emptyset$.

In order to investigate differential equations described by operators of different orders we shall consider more general notion of differential equations. Namely, by a (regular) differential equation we mean a sequence $\mathcal{E} = \{\mathcal{E}_l\}_{l \geq -1}$ of submanifolds $\mathcal{E}_l \subset J^l M$ with $\mathcal{E}_{-1} = M$, $\mathcal{E}_0 = J^0 M = M \times \mathbb{R}$ such that the following conditions hold:

1. $\pi_{l,l-1} : \mathcal{E}_l \to \mathcal{E}_{l-1}$ are bundle projections for all $l \geq 0$.
2. The first prolongations $\mathcal{E}_l^{(1)}$ are smooth submanifolds and $\mathcal{E}_{l+1} \subset \mathcal{E}_l^{(1)}$.

Due to Cartan-Kuranishi theorem on prolongations there exists a (minimal) number l_1 such that $\mathcal{E}_l^{(1)} = \mathcal{E}_{l+1}$ for all $l \geq l_1$. Denote by l_0 the maximal number such that $\mathcal{E}_l = J^l M$ for all $l < l_0$. We call the pair (l_0, l_1) the *order* of $\mathcal{E}$.

We call the equation $\mathcal{E}$ *formally integrable* if all $\mathcal{E}_l$ are non-empty.

1.2. Cartan distribution and metasymplectic structure

Fix a point $x_k \in J^k M$. Then tangent plane to jet-sections span a subspace

$$\mathcal{C}(x_k) = \langle \cup (j_k f)_* \tau_x \mid [f]_x^k = x_k \rangle,$$

which is called the *Cartan subspace* ([KLV]). The corresponding distribution $\mathcal{C}(x_k) \subset T_{x_k} J^k M$ on $J^k M$ is called the *Cartan distribution*.

In J^1M this distribution is nothing else but the standard contact structure. For general k we can describe $\mathcal{C}(x_k)$ in local coordinates as follows.

Every (local) coordinate system (x^i) on M induces coordinates (x^i, p_σ) on J^kM (multiindex σ has length $|\sigma| \le k$), where $p_\sigma\Big([f]_x^k\Big) = \dfrac{\partial^{|\sigma|}f}{\partial x^\sigma}(x)$. Then the differential forms $\omega_\sigma = dp_\sigma - \sum \omega_{\sigma+1_i}dx^i$ span the annulator of $\mathcal{C}$, i.e. $\mathcal{C} = \mathrm{Ker}\{\omega_\sigma\}_{0\le|\sigma|<k}$.

To describe a basis of sections of $\mathcal{C}$ we recall ([KLV]) the *operator of total derivative* $\mathcal{D} : C^\infty(J^kM) \to \Omega^1(M) \otimes_{C^\infty(M)} C^\infty(J^{k+1}M)$. To define $\mathcal{D}$ we note that every function on J^kM is a differential operator of order k. Composing it with a vector field $X \in \mathcal{D}(M)$ we get a differential operator of order $k+1$ producing the needed operator $\mathcal{D}_X : C^\infty(J^kM) \to C^\infty(J^{k+1}M)$.

In coordinates $\mathcal{D}_i = \mathcal{D}_{\partial_{x^i}}$ is given by infinite series

$$\mathcal{D}_i = \partial_{x^i} + \sum p_{\sigma+1_i}\partial_{p_\sigma}. \tag{1}$$

If in the above sum we restrict $|\sigma| < k$ we get vector fields $\mathcal{D}_i^{(k)}$ on J^kM. In terms of them the Cartan distribution on J^kM is given by

$$\mathcal{C} = \langle \mathcal{D}_i^{(k)}, \partial_{p_\sigma}\rangle_{1\le i\le n,|\sigma|=k} \tag{2}$$

Recall that for any distribution Π its curvature $\Xi_\Pi \in \Lambda^2\Pi^* \otimes \nu$ is the 2-form on Π with values in the normal bundle $\nu = TM/\Pi$, defined as $\Xi_\Pi(X,Y) = [X,Y] \bmod \Pi$. It is however more convenient to consider smaller space of values, namely the linear span of the $\mathrm{Im}\,\Xi_\Pi \subset \nu$. We will consider this space as ν.

The curvature of the Cartan distribution is called *metasymplectic structure*. In this case $\nu \simeq S^{k-1}\tau^* (= F(x_{k-2}))$ and so the metasymplectic structure can be viewed as the following 2-form

$$\Omega_k = \Omega_{x_k} \in \Lambda^2(\mathcal{C}^*(x_k)) \otimes S^{k-1}\tau_x^*.$$

Thus for every $\lambda \in S^{k-1}\tau_x$ the evaluation $\Omega_k(\lambda) = \langle \Omega_{x_k}, \lambda\rangle$ is an ordinary 2-form on the Cartan space $\mathcal{C}(x_k)$. In particular for $k = 1$ we have the standard symplectic form Ω_1 on $\mathcal{C}(x_1)$.

Fix some $f \in C^\infty(M)$ with $[f]_x^k = x_k$. Denote $H(x_k) = (j_kf)_*\tau_x$. Then we have a decomposition

$$\mathcal{C}(x_k) = H(x_k) \oplus F(x_{k-1}) \simeq \tau_x \oplus S^k\tau_x^*. \tag{3}$$

Note that $\Omega_k|_H = 0$ for this choice of H because $j_kf(M)$ is an integral manifold of the Cartan distribution. On the other hand it is obvious that $\Omega_k|_F = 0$.

So to calculate Ω_k it is enough to know the value on the bivector $X \wedge \theta$ with $\theta \in S^k\tau_x^*$, $X \in H(x_k) \simeq \tau_x$. This can be expressed in terms of the Spencer operator $\delta : S^k\tau^* \to \tau^* \otimes S^{k-1}\tau^*$ as follows ([Ly]):

$$\Omega_k(\lambda)(X,\theta) = \langle \lambda, i_X\delta\theta\rangle. \tag{4}$$

Let us write the metasymplectic structure is coordinates. The commutators of the basic fields (2) are:

$$[\mathcal{D}_i,\mathcal{D}_j]=0,\ [\partial_{p_\sigma},\partial_{p_\gamma}]=0,\ [\partial_{p_\sigma},\mathcal{D}_i]=\partial_{p_\theta}\ \text{if}\ \sigma=\theta+1_i\ \text{and 0 otherwise.}$$

So we can take $\nu\simeq\langle\partial_{p_\theta}\rangle_{|\theta|=k-1}$ and the above relations represent the metasymplectic structure $\Omega_k=\Xi_{\mathcal{C}}$ on J^kM. Another form of it is

$$\Omega_k=\sum_{|\sigma|=k-1}d\omega_\sigma\otimes\partial_{p_\sigma}.$$

If $\mathcal{E}$ is a PDE represented by submanifolds $\mathcal{E}_k\subset J^kM$ then the restriction of the Cartan distribution on $\mathcal{E}_k$

$$\mathcal{C}_{\mathcal{E}_k}(x_k)=\mathcal{C}(x_k)\cap T_{x_k}(\mathcal{E}_k)$$

is called Cartan distribution on $\mathcal{E}_k$. The restriction of the metasymplectic structure to $\mathcal{C}_{\mathcal{E}_k}$ will be denoted by $\Omega_{\mathcal{E}_k}$.

1.3. Symbols and Spencer cohomologies

The *symbol* of differential equation $\mathcal{E}$ at a point $x_k\in\mathcal{E}_k$ is the vertical tangent space to $\mathcal{E}_k$:

$$g_k(x_k)=T_{x_k}\left(F(x_{k-1})\right)\cap T_{x_k}(\mathcal{E})\subset S^k\tau_x^*.$$

Note that the metasymplectic structure of equation $\mathcal{E}$ takes values in symbols:

$$\Omega_{\mathcal{E}_k}\in\Lambda^2\left(\mathcal{C}^*_{\mathcal{E}_k}(x_k)\right)\otimes g_{k-1}(x_{k-1}).$$

The symbol of prolongations $\mathcal{E}_k^{(1)}$ at the point $x_{k+1}\in F(x_k)$ is a subspace

$$g_k^{(1)}(x_k)=\{\,\theta\in S^{k+1}\tau_x^*\,|\,\delta\theta\in\tau_x^*\otimes g_k(x_k)\},$$

where $\delta:\Lambda^i\tau_x^*\otimes S^{j+1}\tau_x^*\to\Lambda^{i+1}\tau_x^*\otimes S^j\tau_x^*$ is the Spencer δ-operator (one can think of δ as of de Rham operator on forms with polynomial coefficients).

Then for any sequence $\{x_l\in\mathcal{E}_l\,|\,\pi_{l,l-1}(x_l)=x_{l-1}\}$ one gets $\delta:g_l(x_l)\to\tau_x^*\otimes g_{l-1}(x_{l-1})$ and therefore the sequence of symbols $\{g_l(x_l)\}$ defines the Spencer δ-complex

$$0\to g_k(x_k)\xrightarrow{\delta}\tau_x^*\otimes g_{k-1}(x_{k-1})\xrightarrow{\delta}\cdots\xrightarrow{\delta}\Lambda^n\tau_x^*\otimes g_{k-n}(x_{k-n})\to 0,$$

where $n=\dim M$.

The cohomology group at the term $\Lambda^i\tau_x^*\otimes g_j(x_j)$ is called the *Spencer δ-cohomology group* at the point x_k and shall be denoted by $H^{j,i}(\mathcal{E};x_k)$.

1.4. Weyl tensor

A subspace $H \subset T_{x_k}\mathcal{E}_k$ is called *horizontal* if $(\pi_k)_* : H \to T_x M$ is an isomorphism.

Definition 1. *A horizontal distribution $H(x_k)$ on $\mathcal{E}_k$ is called a* Cartan connection *if $H(x_k) \subset \mathcal{C}_{\mathcal{E}_k}(x_k)$.*

One can prove that near every point x_k there exists a Cartan connection H. Any such connection determines the following splitting:

$$\mathcal{C}_{\mathcal{E}_k}(x_k) = H(x_k) \oplus g_k(x_k). \tag{5}$$

Definition 2. *The tensor $\Omega_H(x_k) = \Omega_{x_k}|_{H(x_k)}$ is called the* curvature *of the Cartan connection H at the point $x_k \in \mathcal{E}_k$.*

This curvature due to isomorphism $(\pi_k)_* : H(x_k) \simeq \tau_x$ can be viewed as a 2-form

$$\Omega_H(x_k) \in \Lambda^2(\tau_x^*) \otimes g_{k-1}(x_{k-1}).$$

Note that calculation of Ω_k in decomposition (5) is different of that in a *special* decomposition (3). However (4) produces a way to calculate Ω_H.

Proposition 1 ([Ly]). (i). *The curvature form of the Cartan connection $\Omega_H \in \Lambda^2(\tau^*) \otimes g_{k-1}(x_{k-1})$ is δ-closed.*
(ii). *If H' is another Cartan connection then $\Omega_{H'} = \Omega_H + \delta\sigma$, where $\sigma \in \tau_x^* \otimes g_k(x_k)$ represents H' in decomposition (5): $H' = \mathrm{graph}\{\sigma : H \to g_k\}$.*

So we get a canonical cohomology class:

Definition 3. *The* Weyl tensor *of differential equation $\mathcal{E}$ at a point $x_k \in \mathcal{E}_k$ is the δ-cohomology class*

$$W_k(\mathcal{E}; x_k) = \Omega_H \mod \delta(\tau_x^* \otimes g_k(x_k)) \in H^{k-1,2}(\mathcal{E}; x_k).$$

Theorem 2 ([Ly]). *Let $\pi_{k,k-1} : \mathcal{E}_k \to \mathcal{E}_{k-1}$ be a smooth bundle. Then differential equation $\mathcal{E}_k \subset J^k M$ admits the first prolongation $\mathcal{E}_k^{(1)}$ at a point x_k if and only if $W_k(\mathcal{E}_k; x_k) = 0$.*

So if $W_k(\mathcal{E}) = 0$ we consider the Weyl tensor $W_{k+1}(\mathcal{E}) \in H^{k,2}(\mathcal{E})$ for the equation $\mathcal{E}^{(1)}$, study the equation $W_{k+1}(\mathcal{E}) = 0$ etc.

Remark 3. *Due to Poincaré δ-lemma after some number r of prolongations the second Spencer δ-cohomologies vanish, $H^{k-1+i,2}(\mathcal{E}) = 0$, $i \geq r$. Therefore the conditions $W_k = 0, \dots, W_{k+r-1} = 0$, together with regularity conditions that $g^{(i)}(x_k) \subset S^{k+i}\tau_x^*$ form vector bundles over $\mathcal{E}$ for $0 \leq i \leq r$, guarantee the formal integrability of $\mathcal{E}$. This generalizes classical approach to integrability by Spencer-Quillen-Goldschmidt [G].*

Remark 4. *If $g(x_k) = 0$ the equation $W_k(\mathcal{E}) = 0$ is exactly the Frobenius theorem for the horizontal (in this case) distribution $\mathcal{C}_{\mathcal{E}}$. More generally if $\mathcal{E}$ is the equation of finite type, i.e. $g^{(r)} = 0$ for some r, then the conditions $W_k = 0, \dots, W_{k+r-1} = 0$ together with regularity conditions guarantee the existence of smooth (local) solutions.*

1.5. Example

Consider a PDE $\mathcal{E} \subset J^1(M^n)$ of $\operatorname{codim} \mathcal{E} = n$:

$$\begin{cases} p_1 &= f_1(x^1, \dots, x^n, u) \\ & \dots\dots \\ p_n &= f_n(x^1, \dots, x^n, u). \end{cases} \tag{6}$$

The symbol vanishes $g(x_1) = 0$ and the Spencer sequence is $0 \to \Lambda^2 \tau_x^* \to 0$. Thus $H^{0,2}(\mathcal{E}) = \Lambda^2 \tau^*$, $H^{i,2}(\mathcal{E}) = 0$, $i > 0$. Cartan connection H has a basis

$$\nabla_1 = \partial_{x^1} + p_1 \partial_u + \sum_{j=1}^n \alpha_{1j} \partial_{p_j}, \dots, \nabla_n = \partial_{x^n} + p_n \partial_u + \sum_{j=1}^n \alpha_{nj} \partial_{p_j}.$$

The conditions $\nabla_i \in T_{x_1} \mathcal{E}$, i.e. $\nabla_i(p_j - f_j) = 0$, give

$$\alpha_{ij} = \frac{\partial f_j}{\partial x^i} + p_i \frac{\partial f_j}{\partial u}.$$

Since the symbol is 0, we have $W_1(\mathcal{E}; x_1) = \Omega_1|_H = \sum_{i<j} \Omega_1(\nabla_i, \nabla_j) dx^i \wedge dx^j$. Using $\Omega_1 = dx \wedge dp$ we get $\Omega_1(\nabla_i, \nabla_j) = \alpha_{ji} - \alpha_{ij}$. Since $W_r(\mathcal{E}) = 0$ for $r > 1$, integrability criterion $W_1 = 0$ yields compatibility condition for the system (6):

$$\frac{\partial f_j}{\partial x^i} - \frac{\partial f_i}{\partial x^j} + f_i \frac{\partial f_j}{\partial u} - f_j \frac{\partial f_i}{\partial u} = 0, \quad 1 \le i < j \le n.$$

2. PDEs over 2-dimensional manifolds

In this chapter we investigate formal and smooth integrability of differential equations over 2-dimensional manifolds, $\dim M = 2$. The peculiarity of this case is that the Spencer sequence has (due to dimensional reasons) only 3 terms:

$$0 \to g_{i+1} \to \tau^* \otimes g_i \xrightarrow{\delta} \Lambda^2 \tau^* \otimes g_{i-1} \to 0. \tag{7}$$

2.1. Differential equations of the second order

At first, we consider second order PDE of codimension 2: $\mathcal{E}_2 \subset J^2 M$, $\mathcal{E}_{2+i} = \mathcal{E}_2^{(i)}$, $\mathcal{E}_1 = J^1 M$. In this case $\dim g_2(x_2) = 1$.

Recall that a non-zero covector $p \in \tau_x^*$ is called (real) *characteristic* at $x_2 \in \mathcal{E}$ if $p^2 \in g_2(x_2)$. The first Spencer δ-sequence is as follows:

$$0 \to g_2^{(1)}(x_2) \to \tau_x^* \otimes g_2(x_2) \xrightarrow{\delta} \Lambda^2 \tau_x^* \otimes \tau_x^* \to 0. \tag{8}$$

Proposition 3. *A differential equation $\mathcal{E}_2 \subset J^2 M$ of codimension 2 has no characteristics at a point $x_2 \in \mathcal{E}_2$ if and only if the map δ in (8) is surjective. In this case $W_2(\mathcal{E}_2; x_2) = 0$.*

Proof. The dimensions of both sides of δ equal 2. So $\operatorname{Ker}\delta \neq 0 \Leftrightarrow \operatorname{CoKer}\delta \neq 0$. Now if $p \in \tau_x^*$ is a characteristic covector then $p \otimes p^2 \in \tau_x^* \otimes g_2(x_2) \in \operatorname{Ker}\delta$ or equivalently $p^3 \in g_2^{(1)}(x_2)$.

On the other hand if δ is not surjective then there is a proper subspace $U^* \subset \tau^*$ such that $g \subset S^2U^*$. Every covector of U^* is characteristic.

So surjectivity of δ is equivalent to $g_2^{(1)}(x_2) = 0$ and to $H^{1,2}(\mathcal{E}_2; x_2) = 0$. $\square$

Now from the results of §1.4 we deduce

Proposition 4. *Let $\mathcal{E}_2 \subset J^2M$ has no real characteristics at all points $x_2 \in \mathcal{E}_2$. Then the 1-st prolongation $\mathcal{E}_3 = \mathcal{E}_2^{(1)}$ exists and $\pi_{3,2} : \mathcal{E}_3 \to \mathcal{E}_2$ is a diffeomorphism. Moreover $g_2^{(i)}(x_2) = 0$ for all $i \geq 1$.* $\square$

From the higher Spencer sequences (7) we get that $H^{2,2}(\mathcal{E}_2) = \Lambda^2\tau_x^* \otimes g_2$ is one-dimensional and $H^{i,2}(\mathcal{E}) = 0$ for $i > 2$.

So the (scalar-valued) 2-form $W_3 = \Omega_3$ is the only obstruction to formal integrability of $\mathcal{E}$. Because the equation has finite type, the condition $W_3 = 0$ implies smooth solvability.

2.2. Local coordinates

Let the equation be given by relations $F = G = 0$, i.e. the submanifold $\mathcal{E}_2$ is

$$\begin{cases} F(x_1, x_2, u, p_1, p_2, p_{11}, p_{12}, p_{22}) = 0, \\ G(x_1, x_2, u, p_1, p_2, p_{11}, p_{12}, p_{22}) = 0. \end{cases}$$

Regularity of the equation is equivalent to linear independence of the vertical differentials $d^vF = \sum F_{ij}dp_{ij}$ and $d^vG = \sum G_{ij}dp_{ij}$, where $F_{ij} = \dfrac{\partial F}{\partial p_{ij}}$.

Theorem 5. (i). *The differential equation $\mathcal{E}_2$ has no real characteristics if*

$$-\varpi = \det\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{12} & \rho_{22}\end{pmatrix} \neq 0, \ \textit{where}$$

$$\rho_{11} = F_{12}G_{22} - F_{22}G_{12},\ \rho_{12} = F_{22}G_{11} - F_{11}G_{22},\ \rho_{22} = F_{12}G_{12} - F_{12}G_{11}$$

are coefficients of the decomposition of $d^vF \wedge d^vG$ in the basic $dp_{ij} \wedge dp_{kl}$.

(ii). *A basis vector in the 1-dimensional vector space $g_2(x_2)$ is*

$$e_g = \rho_{11}\partial_{p_{11}} + \rho_{12}\partial_{p_{12}} + \rho_{22}\partial_{p_{22}}.$$

(iii). *The Weyl tensor W_3 is equal to*

$$W_3 = \frac{1}{\varpi}\,[F,G]|_{\mathcal{E}^{(1)}}\ dx^1 \wedge dx^2 \otimes e_g,$$

where

$$\begin{aligned}[F,G] = {} & \mathcal{D}_1^2(F)G_{11} + \mathcal{D}_{12}(F)G_{12} + \mathcal{D}_2^2(F)G_{22}\\ & -\mathcal{D}_1^2(G)F_{11} - \mathcal{D}_{12}(G)F_{12} - \mathcal{D}_2^2(G)F_{22}.\end{aligned}$$

Remark 5. *The bracket above is a function on J^3M but due to isomorphism $\pi_{3,2} : \mathcal{E}^{(1)} \to \mathcal{E}$ the restriction $[F,G]|_{\mathcal{E}^{(1)}}$ can be considered as a function on $\mathcal{E}$. Note also that in the definition of $[,]$ the total derivative operators can be changed by $\mathcal{D}_i^{(3)}$, which are the following vector fields on J^3M:*

$$\begin{aligned}
\mathcal{D}_1^{(3)} &= \partial_{x^1} + p_1\partial_u + p_{11}\partial_{p_1} + p_{12}\partial_{p_2} + p_{111}\partial_{p_{11}} + p_{112}\partial_{p_{12}} + p_{122}\partial_{p_{22}}, \\
\mathcal{D}_2^{(3)} &= \partial_{x^2} + p_2\partial_u + p_{12}\partial_{p_1} + p_{22}\partial_{p_2} + p_{112}\partial_{p_{11}} + p_{122}\partial_{p_{12}} + p_{222}\partial_{p_{22}}.
\end{aligned}$$

Proof. The equation of $\mathcal{E}^{(1)}$ is $\{F = G = \mathcal{D}_1F = \mathcal{D}_2F = \mathcal{D}_1G = \mathcal{D}_2G = 0\}$. Since the symbol of $\mathcal{E}^{(1)}$ vanishes, the Cartan distribution is horizontal and generated by the vector fields of the form

$$\nabla_i = \mathcal{D}_i^{(3)} + \alpha_i\partial_{p_{111}} + \beta_i\partial_{p_{112}} + \gamma_i\partial_{p_{122}} + \delta_i\partial_{p_{222}}, \quad i = 1, 2.$$

The conditions $\nabla_i \in \mathcal{C}_{\mathcal{E}^{(1)}}$ are equivalent to $\nabla_i(\mathcal{D}_jF) = 0, \nabla_i(\mathcal{D}_jG) = 0$, so that we get a system of linear equations for $\alpha_i, \beta_i, \gamma_i, \delta_i$ on $\mathcal{E}^{(1)}$:

$$\begin{cases}
\alpha_iF_{11} + \beta_iF_{12} + \gamma_iF_{22} = -\mathcal{D}_i^{(3)}\mathcal{D}_1F \\
\beta_iF_{11} + \gamma_iF_{12} + \delta_iF_{22} = -\mathcal{D}_i^{(3)}\mathcal{D}_2F \\
\alpha_iG_{11} + \beta_iG_{12} + \gamma_iG_{22} = -\mathcal{D}_i^{(3)}\mathcal{D}_1G \\
\beta_iG_{11} + \gamma_iG_{12} + \delta_iG_{22} = -\mathcal{D}_i^{(3)}\mathcal{D}_2G.
\end{cases}$$

The determinant of this system is exactly ϖ.

The Weyl tensor coincides with the curvature tensor and therefore equals

$$W_3 = d\omega_{11}|_{\mathcal{C}_{\mathcal{E}^{(1)}}} \otimes \partial_{p_{11}} + d\omega_{12}|_{\mathcal{C}_{\mathcal{E}^{(1)}}} \otimes \partial_{p_{12}} + d\omega_{22}|_{\mathcal{C}_{\mathcal{E}^{(1)}}} \otimes \partial_{p_{22}}.$$

It is easy to see that

$$d\omega_{11}(\nabla_1, \nabla_2) = \alpha_2 - \beta_1,\ d\omega_{12}(\nabla_1, \nabla_2) = \beta_2 - \gamma_1,\ d\omega_{22}(\nabla_1, \nabla_2) = \gamma_2 - \delta_1.$$

From the above linear system one gets

$$\alpha_2 - \beta_1 = \frac{\rho_{11}}{\varpi}[F,G], \quad \beta_2 - \gamma_1 = \frac{\rho_{12}}{\varpi}[F,G], \quad \gamma_2 - \delta_1 = \frac{\rho_{22}}{\varpi}[F,G].$$

Therefore $\varpi W_3 = [F,G]|_{\mathcal{E}^{(1)}}\ dx^1 \wedge dx^2 \otimes e_g$. □

2.3. Mayer brackets

At first we shall fix the expression for the Weyl tensor which we got in local coordinates. The definition is general and we suppose $\dim M = n \geq 2$ in this section only. As before a point $x_k \in J^kM$ is presented by coordinates (x^i, p_σ) with $i = 1, \ldots, n$ and the multiindex $\sigma = (i_1, \ldots, i_n)$ of length $|\sigma| = \sum i_s \leq k$.

Below we consider higher order operators of total derivative $\mathcal{D}_\sigma = \mathcal{D}_1^{i_1} \ldots \mathcal{D}_n^{i_n}$ with $\mathcal{D}_j$ given by (1). These operators $\mathcal{D}_\sigma : C^\infty(J^lM) \to C^\infty(J^{l+|\sigma|}M)$ have order $|\sigma|$.

Definition 4. *The* (higher) Mayer bracket *of two functions* $F \in C^\infty(J^k M)$, $G \in C^\infty(J^l M)$ *is the following function on* $J^{k+l}M$*:*

$$[F,G] = \sum_{|\sigma|=l} \mathcal{D}_\sigma(F) G_\sigma - \sum_{|\tau|=k} \mathcal{D}_\tau(G) F_\tau, \qquad \textit{where } F_\sigma = \frac{\partial F}{\partial p_\sigma}. \tag{9}$$

This bracket is bilinear, skew-symmetric, but the Jacobi identity is violated. For $k=1$ it satisfies additionally Leibniz rule and can be described as follows. If $F,G \in C^\infty(J^1M)$ are two functions on the space of 1st jets, then the bracket is again a function on J^1M given by $[F,G] = \Omega_1(X_F, X_G)$. Here Ω_1 is the canonical symplectic form and X_F is the contact vector field determined by F.

Lemma 6. *The Mayer bracket* $[F,G]$ *of two functions* $F \in C^\infty(J^kM)$*,* $G \in C^\infty(J^lM)$ *is a function lifted from* $J^{k+l-1}M$ *by means of the map* $\pi_{k+l,k+l-1}$*.*

Proof. Actually in the decomposition by p_θ the higher order terms in both sums of (9) equal $\sum_{|\tau|=k} \sum_{|\sigma|=l} F_\tau G_\sigma p_{\sigma\tau}$ and so they sum to zero. □

Recall that the higher Jacobi bracket of two functions $F \in C^\infty(J^kM)$, $G \in C^\infty(J^lM)$ is given in coordinates by the following expression:

$$\{F,G\} = \sum_{0\le|\sigma|\le l} \mathcal{D}_\sigma(F) G_\sigma - \sum_{0\le|\tau|\le k} \mathcal{D}_\tau(G) F_\tau. \tag{10}$$

See [KLV] for coordinates-free definition.

To get invariant meaning of the Mayer bracket we define

$$\mathcal{J}^{(k,l)}(F,G) = \Big\langle \mathcal{D}_\sigma(F), \mathcal{D}_\tau(G) \Big\rangle_{0\le|\tau|<k,\, 0\le|\sigma|<l}$$

to be the ideal in $C^\infty(J^{k+l-1}M)$ generated by F, G and their total derivatives up to the orders $(l-1)$ and $(k-1)$ respectively.

Proposition 7. *Two brackets agree modulo the above ideal:*

$$[F,G] = \{F,G\} \bmod \mathcal{J}^{(k,l)}(F,G). \qquad \square$$

As a corollary we observe that the restriction of the Mayer bracket to the prolongation $\mathcal{E}_{k+l-1}$ does not depend on a choice of coordinates, if the latter is a smooth submanifold in $J^{k+l-1}M$.

We can reformulate now theorem 5 in terms of Jacobi brackets:

Theorem 8. *The regular system* $F=0$*,* $G=0$ *of 2-nd order differential equations which has no characteristics is formally and smooth integrable iff* $\{F,G\} = 0 \bmod \mathcal{J}^{(2,2)}(F,G)$*.*

2.4. Symbols of higher order differential equations

We investigate here the case of differential equation given by two scalar differential operators $F \in C^\infty(J^k M)$ and $G \in C^\infty(J^l M)$ of orders $2 \le k \le l$ over 2-dimensional manifold M^2.

In this case, $\mathcal{E}_i = J^i M$ for $i \le k-1$, $\mathcal{E}_k = F^{-1}(0) \subset J^k M$, $\mathcal{E}_{k+j} = \mathcal{E}_k^{(j)} \subset J^{k+j} M$ for $0 \le j \le l-k-1$, $\mathcal{E}_l = \mathcal{E}_k^{(l-k)} \cap G^{-1}(0) \subset J^l M$ and $\mathcal{E}_{l+s} = \mathcal{E}_l^{(s)}$ for $s \ge 0$.

To find symbols $g_r = g_r(x_r)$, $r \ge k$, of the equation at the point $x_r \in \mathcal{E}_r$ we denote by $\phi_j : S^{k+j}\tau_x^* \to S^j \tau_x^*$, $j \ge 0$, the prolongations of the symbol ϕ_0 of the operator F and by $\gamma_j : S^{l+j}\tau_x^* \to S^j\tau_x^*$ the corresponding prolongations of the symbol γ_0 of G at the corresponding point x_r. We have:

$$g_i = \begin{cases} S^i \tau_x^*, & \text{when } i \le k-1, \\ \operatorname{Ker}\phi_{i-k}, & \text{when } k \le i \le l-1, \\ \operatorname{Ker}\phi_{i-k} \cap \operatorname{Ker}\gamma_{i-l}, & \text{when } i \ge l. \end{cases}$$

Lemma 9. *Let $\zeta_1, \dots, \zeta_k \in \tau_x^*$ be non-zero covectors such that $\zeta_i \wedge \zeta_j \ne 0$ when $i \ne j$. Then the tensors*

$$\zeta_1^r, \dots, \zeta_k^r \in S^r \tau_x^*$$

generate $S^r\tau_x^*$ if $k \ge r+1$ and are linear independent if $k \le r+1$.

Proof. Let us choose $x = \zeta_1$ and $y = \zeta_2$ as a basis in τ_x^*. Then $\zeta_i = x + \lambda_i y$ for $i \ge 3$ and $\lambda_i \ne \lambda_j$ if $i \ne j$.

Consider a linear relation

$$c_1 x^r + c_2 y^r + \sum_{i=3}^{k} c_i (x + \lambda_i y)^r = b \in S^r \tau_x^*.$$

In the basis $e_s = \frac{r!}{(r-s)!s!} x^{r-s} y^s$, $s = 0, \dots, k$, this relation can be rewritten as a linear system of equations

$$\begin{cases} c_1 + \sum_3^k c_i \quad = b_0, \\ \sum_3^k c_i \lambda_i^s = b_s, \quad \text{when } 1 \le s \le r-1, \\ c_2 + \sum_3^k c_i \lambda_i^r = b_r. \end{cases}$$

The middle system has solutions if $k-2 \ge r-1$, has a unique solution for every b if $k-2 = r-1$ (Vandermonde determinant) and has only the trivial solution $c_s = 0$ if $k-2 \le r-1$ and $b_s = 0$. $\square$

Proposition 10. *Let $\phi_0 \ne 0$. Then the prolongations ϕ_j are epimorphisms and so* $\dim \operatorname{Ker}\phi_j = k$.

Proof. For any covector ζ one has:

$$\phi_j(\zeta^{k+j}) = (k+j)\cdots(k+1)\phi_0(\zeta^k)\zeta^j.$$

Therefore, due to the above lemma, tensors $\phi_j(\zeta_1^{k+j}), \dots, \phi_j(\zeta_{j+1}^{k+j})$ form a basis of S^jT^* for any sequence of covectors $\zeta_1, \dots, \zeta_{j+1}$ that satisfy the conditions of the lemma and, in addition, $\phi_0(\zeta_i^k) \neq 0$ for all $i \leq j+1$. □

Recall that a non-zero covector $\zeta \in \tau_x^*$ such that $\phi_0(\zeta^k) = 0$ is called *characteristic* (for ϕ_0). Therefore, to get basis into $\operatorname{Im}\phi_j$ we could consider images of degrees of non-characteristic covectors.

Corollary. *Assume that ϕ_0 has k distinct characteristics $\zeta_1, \dots, \zeta_k$. Then $\zeta_1^{k+j}, \dots, \zeta_k^{k+j}$ form a basis of* $\operatorname{Ker}\phi_j \subset S^{k+j}\tau_x^*$. □

Remark 6. *The result with obvious correction* $\operatorname{Im}\zeta_s^{k+j}$ *and* $\operatorname{Re}\zeta_s^{k+j}$ *instead of* ζ_s^{k+j} *holds for complex characteristics too.*

2.5. Characteristics and Spencer cohomologies

Proposition 11. *Let the symbol $\phi_0 : S^k\tau_x^* \to \mathbb{R}$ have distinct characteristics, say $\zeta_1, \dots, \zeta_k$ and let $\gamma_0 : S^l\tau_x^* \to \mathbb{R}$, $l \geq k$, be another symbol such that $\gamma_0(\zeta_r^l) \neq 0$ for all $r = 1, \dots, k$. Then*

$$\begin{aligned}
\dim\operatorname{Ker}\phi_i &= k, && \text{for } i \geq 0,\\
\dim\operatorname{Ker}\gamma_i &= l, && \text{for } i \geq 0,\\
\dim(\operatorname{Ker}\phi_i \cap \operatorname{Ker}\gamma_{k+i-l}) &= l-i-1, && \text{for } l-k \leq i \leq l-1,\\
\dim(\operatorname{Ker}\phi_i \cap \operatorname{Ker}\gamma_{k+i-l}) &= 0, && \text{for } i \geq l-1.
\end{aligned}$$

Proof. Consider the restrictions

$$\gamma_{k+i-l} : \operatorname{Ker}\phi_i \to S^{k+i-l}\tau_x^*.$$

The basis vectors $\zeta_1^{k+i}, \dots, \zeta_k^{k+i}$ of $\operatorname{Ker}\phi_i$ map to

$$(k+i)\cdots(l+1)\gamma_0(\zeta_1^l)\zeta_1^{k+i-l}, \dots, (k+i)\cdots(l+1)\gamma_0(\zeta_k^l)\zeta_k^{k+i-l}$$

and therefore generate the whole $S^{k+i-l}\tau_x^*$ if $k \geq k+i-l+1$.

In this case

$$\dim(\operatorname{Ker}\phi_i \cap \operatorname{Ker}\gamma_{k+i-l}) = \dim\operatorname{Ker}\phi_i - (k+i-l+1) = l-i-1.$$

□

Note that the proof is valid if we assume γ_0 has distinct characteristics and ϕ_0 does not vanish on the corresponding prolongations. In this case we consider restrictions of ϕ_i to $\operatorname{Ker}\gamma_j$ and get the following

Proposition 12. *Let the symbol $\gamma_0 : S^l\tau_x^* \to \mathbb{R}$ have distinct characteristics, say $\zeta_1,\dots,\zeta_l$ and let $\phi_0 : S^k\tau_x^* \to \mathbb{R}$, $k \le l$ be another symbol such that $\phi_0(\zeta_r^k) \neq 0$ for all $r = 1,\dots,l$. Then*

$$\begin{aligned}
\dim \operatorname{Ker} \gamma_i &= l, && \textit{for } i \ge 0,\\
\dim \operatorname{Ker} \phi_i &= k, && \textit{for } i \ge 0,\\
\dim(\operatorname{Ker} \phi_i \cap \operatorname{Ker} \gamma_{k+i-l}) &= l-i-1, && \textit{for } l-k \le i \le l-1,\\
\dim(\operatorname{Ker} \phi_i \cap \operatorname{Ker} \gamma_{k+i-l}) &= 0, && \textit{for } i \ge l-1.
\end{aligned}$$

Theorem 13. *Let $\phi_0 : S^k\tau_x^* \to \mathbb{R}$ and $\gamma_0 : S^l\tau_x^* \to \mathbb{R}$, $k \le l$ have no common characteristics and either ϕ or γ have distinct characteristics. Let $g_i = \operatorname{Ker} \phi_{i-k} \cap \operatorname{Ker} \gamma_{i-l}$ be the symbols defined by ϕ_0 and γ_0. Then*

1. $$\dim g_i = \begin{cases} i+1, & \textit{if } 0 \le i < k,\\ k, & \textit{if } k \le i \le l-1,\\ k+l-1-i, & \textit{if } l \le i \le k+l-1,\\ 0, & \textit{if } i \ge k+l-1.\end{cases}$$

2. $$\dim H^{i,2}(g) = \begin{cases} 1, & i = k+l-2,\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. Part 1 is a consequence of propositions 11 and 12. By the construction $g_{i+1} = g_i^{(1)}$ if $k \le i \le l-2$ or $i \ge l$.

Consider the Spencer δ-complex (7). If $k \le i \le l-2$ then, due to dimensional count of part 1, the Euler characteristic of the complex equals 0. Moreover, $g_{i+1} = g_i^{(1)}$ and hence $H^{i,1}(g) = 0$. Therefore $H^{i-1,2}(g) = 0$.

For the same reasons, $H^{i-1,2}(g) = 0$ when $l+1 \le i \le k+l-2$ and trivially when $i \ge k+l$. So we should investigate the following three cases:

A. $i = l-1$,

B. $i = l$,

C. $i = l+k-1$.

For the case A $H^{l-2,2}(g) = H^{l-2,2}(\operatorname{Ker} \phi_0) = 0$. Note that the Euler characteristic $\chi = -1$ in the case $k \neq l$ yields $\dim H^{l-1,1} = 1$.

In the case B the complex takes a form

$$0 \to g_l^{(1)} \to \tau_x^* \otimes g_l \to \Lambda^2\tau_x^* \otimes g_{l-1} \to 0$$

and since the Euler characteristic is 0, we get $H^{l-1,2}(g) = 0$.

Finally, in the case C we have the complex

$$0 \to g_{k+l-1}^{(1)} \to \tau_x^* \otimes g_{k+l-1} \to \Lambda^2\tau_x^* \otimes g_{k+l-2} \to 0.$$

Therefore the Euler characteristic gives $\dim H^{k+l-2,2}(g) = 1$. □

Remark 7. *The symbols $\phi_0 : S^k\tau_x^* \to \mathbb{R}$ and $\gamma_0 : S^l\tau_x^* \to \mathbb{R}$ can be considered as polynomials $P_\phi \in S^k\tau_x$ and $P_\gamma \in S^l\tau_x$. Denote by $R(P_\phi, P_\gamma)$ the resultant of polynomials and by $D(P_\phi)$ the discriminant. Then the conditions of the theorem take the following form*

$$R(P_\phi, P_\gamma) \neq 0 \text{ and either } D(P_\phi) \neq 0 \text{ or } D(P_\gamma) \neq 0.$$

2.6 Calculations in coordinates

In this section we find coordinate expressions for constructions of the previous section. We shall assume that differential operators under consideration satisfy the conditions of theorem 13.

We choose local coordinates x^1, x^2 in M and denote by $p_{i,j}$ the induced coordinates in J^kM, where $p_{i,j}([f]_x^k) = \dfrac{\partial^{i+j} f}{(\partial x^1)^i(\partial x^2)^j}(x)$. In other words $p_{i,j} = p_\sigma$ for multiindex $\sigma = (1,\ldots,1,2,\ldots,2)$ of degree $|\sigma| = i+j$ with i units and j digits two. We also write $F_{i,j} = \dfrac{\partial F}{\partial p_{i,j}}$.

Let us choose the following tensors

$$e_{s,k-s} = \frac{(dx^1)^s(dx^2)^{k-s}}{s!(k-s)!}, \quad s = 0,1,\ldots,k,$$

as a basis of $S^k\tau_x^*$.

The dual basis of $S^k\tau_x$ is $\partial^{i,k-i} = (\partial_{x^1})^i(\partial_{x^2})^{k-i}$.

In these notations the symbols σ_F and σ_G of the differential operators $F \in C^\infty(J^kM)$ and $G \in C^\infty(J^lM)$ are:

$$\sigma_F = \sum_{i=0}^{k} F_{i,k-i}\partial^{i,k-i}, \qquad \sigma_G = \sum_{j=0}^{l} G_{j,l-j}\partial^{j,l-j}.$$

To describe a basis of 1-dimensional space $g_{k+l-2} \subset S^{k+l-2}\tau_x^*$ we make the following elementary observation:

Lemma 14. *In the linear space $\langle e_1,\ldots,e_n\rangle \simeq \mathbb{R}^n$ vector*

$$e = \det \begin{Vmatrix} a_{11} & \ldots & \ldots & a_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1,1} & \ldots & \ldots & a_{n-1,n} \\ e_1 & \ldots & \ldots & e_n \end{Vmatrix}$$

satisfies the following system of linear equations

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n & = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n-1,1}x_1 + \ldots + a_{n-1,n}x_n & = 0. \end{cases}$$

Proposition 15. *The tensor* $e_g =$

$$
= \det \left\| \begin{array}{cccccccc}
F_{k,0} & F_{k-1,1} & \dots & \dots & F_{0,k} & 0 & \dots & 0 \\
0 & F_{k,0} & \ddots & & F_{1,k-1} & F_{0,k} & & \vdots \\
\vdots & \vdots & \ddots & & \dots & F_{1,k-1} & \ddots & 0 \\
0 & 0 & \dots & F_{k,0} & \dots & & \dots & F_{0,k} \\
G_{l,0} & G_{l-1,1} & \dots & G_{2,l-2} & \dots & & \dots & 0 \\
0 & G_{l,0} & \ddots & \vdots & \ddots & G_{0,l} & \ddots & 0 \\
\vdots & \dots & \ddots & & \ddots & & \ddots & 0 \\
0 & 0 & \dots & G_{k,l-k} & \dots & & \dots & G_{0,l} \\
e_{k+l-2,0} & e_{k+l-3,1} & \dots & & e_{l-2,k} & \dots & & e_{0,k+l-2}
\end{array} \right\|
$$

with $(l-1)$ *F-rows and* $(k-1)$ *G-rows gives a basis of* $g_{k+l-2} \subset S^{k+l-2}\tau_x^*$.

Proof. It follows from the above lemma and observation that a tensor

$$\lambda = \sum_{i=0}^{k+l-2} \lambda_{i,k+l-2-i} e_{i,k+l-2-i}$$

belongs to g_{k+l-2} iff

$$
\begin{aligned}
\sum_{i=0}^{k} F_{i,k-i} \lambda_{i+r,k+l-2-i-r} &= 0, \qquad r = 0, \dots, l-2, \\
\sum_{j=0}^{l} G_{j,l-j} \lambda_{j+s,k+l-2-j-s} &= 0, \qquad s = 0, \dots, k-2.
\end{aligned}
$$

□

The following result is the straightforward generalization of the case of 2-nd order differential equations. We again assume the conditions of theorem 13 are fulfilled.

Theorem 16. *For a differential equation* $\mathcal{E}$ *of order* (k,l) *on two-dimensional manifold, given by relations* $\{F = G = 0\}$, *the Weyl tensor* W_{k+l-1} *has the following form:*

$$W_{k+l-1}(\mathcal{E}) = \frac{1}{R(P_\phi, P_\gamma)} \, [F,G]|_{\mathcal{E}_{k+l-1}} \; dx^1 \wedge dx^2 \otimes e_g,$$

where $R(P_\phi, P_\gamma)$ *is the resultant of the polynomials* P_ϕ, P_γ *and*

$$[F,G] = \sum_{i=0}^{k} F_{i,k-i} \mathcal{D}_1^i \mathcal{D}_2^{k-i}(G) - \sum_{j=0}^{l} G_{j,l-j} \mathcal{D}_1^j \mathcal{D}_2^{l-j}(F)$$

is the Mayer bracket of F *and* G.

Remark 8. *For* $k = l = 2$ *we have:* $R(P_\phi, P_\gamma) = \varpi$ *in notations of* §*2.2.*

2.7 Smooth solvability: the main theorem

By the results of §2.3 for differential equations of the considered type the restriction of the Mayer bracket $[,]$ coincides with that of the Jacobi bracket $\{,\}$ on $\mathcal{E}_{k+l-1}$. Therefore the integrability condition $W_{k+l-1}=0$ can be expressed in term of both brackets.

The above considerations with characteristics were valid for $l \geq k \geq 2$, but as easily seen they hold also for the case of PDE of codimension 2 and order $(1,l)$. If $F \in C^{\infty}(J^1M)$ the covectors $p \in g_F(x_1)$ are characteristics and we require for the symbols $(g_F(x_1))^l \not\subset g_G(x_l)$ at any point $x_l \in \mathcal{E}_l$.

Theorem 17. *Assume that differential equations*

$$F(x^1,x^2,u,\dots,p_{k,0},\dots,p_{0,k})=0 \text{ and } G(x^1,x^2,u,\dots,p_{l,0},\dots,p_{0,l})=0$$

have no common characteristics and at least one of them has distinct characteristics.

Consider the system $\mathcal{E}$ given by $\{F=0,G=0\}$. Let

$$[F,G]|_{\mathcal{E}_{k+l-1}}=0.$$

Then for any point $x_{k+l-1}\in\mathcal{E}_{k+l-1}$ there exists a unique smooth solution h of the system such that $x_{k+l-1}=[h]_x^{k+l-1}$.

Proof. The Cartan distribution on $\mathcal{E}_{k+l-1}$ is completely integrable if and only if $W_{k+l-1}=0$. Therefore the result follows from the above and the Frobenius theorem. □

2.8. Characteristic equations

In this section we outline the characteristic case.

Let us begin with second order PDE $\mathcal{E}\subset J^2M$ of $\operatorname{codim}\mathcal{E}=2$ on a two-dimensional manifold. Suppose it is characteristic, i.e. the map δ of (8) is not surjective. Therefore $\dim g^{(i)}=1$ for all $i\geq 0$.

Moreover $g=(U^*)^2$ for some proper subspace $U^*\subset\tau_x^*$ and therefore the algebraic prolongations are $g_i=(U^*)^i\subset S^i\tau_x^*$ and $g_{i+1}=g^{(i)}$. Thus from vanishing of the Euler characteristic of the sequence (7) we deduce $W_i\in H^{i-1,2}(\mathcal{E})=0$ for all $i>2$. The only obstruction for the existence of prolongation $\mathcal{E}^{(1)}$ belongs to the one-dimensional group $H^{1,2}(\mathcal{E})$.

Under some regularity assumptions the PDE becomes of the form $\mathcal{D}_1f=h_1$, $\mathcal{D}_2f=h_2$ for some functions $f,h_1,h_2\in C^{\infty}(J^1M)$. Then the obstruction W_2 is proportional to the difference $\mathcal{D}_1h_2-\mathcal{D}_2h_1$. If $W_2=0$ the equation is formally integrable and so can be prolonged infinitely to $\mathcal{E}^{(\infty)}$. Note however that the equation is of infinite type and only existence of formal solutions can be concluded.

The higher-dimensional version of the above perfectly exists. We consider for simplicity the case $l=k$. So let PDE $\mathcal{E}\subset J^kM$ of $\operatorname{codim}=2$ be given by differential operators $F,G\in C^{\infty}(J^kM)$. The symbol $g\subset S^k\tau^*$ has dimension

$k-1$. Suppose the equation $\mathcal{E}$ is regular at x_k, i.e. the dimensions of $g^{(i)}$ are locally constant.

Proposition 18. *There exists a number $s \in [0, k-1]$ such that*

$$\dim H^{k+i-1,2}(\mathcal{E}) = \begin{cases} 1, & \text{for } i = s, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover

$$\dim g^{(i)} = \begin{cases} k-i-1, & \text{for } 0 \le i \le s, \\ k-s-1, & \text{for } i \ge s. \end{cases}$$

Proof. Actually consider the Spencer sequence

$$0 \to g^{(i+1)} \to \tau^* \otimes g^{(i)} \xrightarrow{\delta_i} \Lambda^2 \tau^* \otimes g^{(i-1)} \to 0. \tag{11}$$

Let's argue inductively starting from $i = 0$. The dimensions $\dim g^{(i-1)} = k - i$, $\dim g^{(i)} = k - i - 1$ and an easy argumentation shows that $\dim \operatorname{Im} \delta \ge k - i - 1$. So two cases are possible.

First the map δ can be surjective (general case). Then $\dim g^{(i+1)} = k - i - 2$ and we consider the following step. The procedure should stop for $i = k - 1$ when $\dim g^{(k-1)} = 0$. Then $g^{(k-2)} \simeq \mathbb{R}$ and the complex (11) shows that $\dim H^{2k-2,2}(\mathcal{E}) = 1$.

Second on some step $i = s$ the differential δ in the complex (11) can have the image of codimension 1. Then this complex has 1-dimensional second cohomology group. Moreover then $\dim g^{(s+1)} = \dim g^{(s)} = 1$. So from the inequality $\dim \operatorname{Im} \delta_i \ge k - s - 1$ for $i \ge s$ we conclude that $\dim g^{(s+i)}$ stabilizes and all the complexes (11) for $i > s$ are exact. The value of $\dim H^{k+i-1,2}(\mathcal{E})$ follows. □

In the case $s < k-1$ the equation is characteristic and of infinite type. Again in a regular case we have the only one condition $W_{k+s} = 0$ for formal (but not smooth) integrability.

3. Applications

Here we show how the generalization of Lagrange-Charpit method works for higher order equations. We consider only examples of the second order equations.

3.1. Examples

In this section we use standard notations $u_{xx} = p_{11}$ etc.

$\underline{\mathbf{1}}^\circ$ The system $\mathcal{E}$

$$\begin{cases} u_{xx} & + u_{yy} & = 0 \\ u_x u_{xy} & + u_y u_{yy} & = 0 \end{cases}$$

is compatible. Actually if $F = p_{11} + p_{22}$, $G = p_1 p_{12} + p_2 p_{22}$, then

$$[F, G]|_{\mathcal{E}} = 4G \frac{p_{11}p_{22} - p_{12}^2}{p_1^2 + p_2^2} = 0.$$

So $\mathcal{E}$ has 4-parametric solution $u(x, y) = e^{\alpha x + \beta} \sin(\alpha y + \gamma) + \delta$.

2° Let us find solution of f-Gordon equation for $f(u) = 2u$ by means of Laplace equation. Actually overdetermine it:

$$\mathcal{E} : \begin{cases} u_{xy} & = 2u \\ u_{xx} + u_{yy} & = 0. \end{cases}$$

One easily checks that the Mayer bracket vanishes, $[F, G] = -2G = 0$. So the system is compatible. To find the solution we consider $\mathcal{E}$ as 6-dimensional manifold with coordinates $(x, y, u, u_x = p_1, u_y = p_2, u_{xx} = p_{11})$ and write the restriction of complete derivatives:

$$\begin{aligned} \xi_x &= \mathcal{D}_x|_{\mathcal{E}} = \partial_x + 2u_x \partial_u + u_{xx}\partial_{u_x} + 2u\partial_{u_y} - 4u_y \partial_{u_{xx}}, \\ \xi_y &= \mathcal{D}_y|_{\mathcal{E}} = \partial_y + 2u_y \partial_u + 2u\partial_{u_x} - u_{xx}\partial_{u_y} + 4u_x \partial_{u_{xx}}. \end{aligned}$$

Vanishing of the Mayer bracket means the involutivity $[\xi_x, \xi_y] = 0$ (one can check it directly). Now we integrate the equation of characteristics $\dot{x} = \xi_x(x)$ and $\dot{y} = \xi_y(y)$ and get the solution

$$u(x, y) = e^{x+y}(c_1 \cos(x-y) + c_2 \sin(x-y)) + e^{-(x+y)}(c_3 \cos(x-y) + c_4 \sin(x-y)).$$

3° Consider PDE $F = u_{xx}u_{yy} - u_{xy}u + cxy = 0$ with $c = \text{const}$. To solve it let us note that $G = u_{xx}$ commutes with F: $[F, G]|_{\mathcal{E}} = 0$. So we can consider the common solution. From the second equation $u = f(y)x + g(y)$ and substituting this into our PDE we get the solution $u(x, y) = \pm x\sqrt{cy^2 + c_1}$.

4° Consider the functions $\phi_2^0 = u_{xx} + uu_x$ and $\phi_2^1 = tu_{xx} + (tu + x/2)u_x + u/2$ on $J^2(\mathbb{R}^2)$. They commute since $[\phi_2^0, \phi_2^1] = \mathcal{D}_x^2(\phi_2^1 - t\phi_2^0)$. Actually this is the 2-dimensional subalgebra of the higher symmetries of the Burgers' equation ([KLV],ch.VIII). Of course these two functions alone are meaningless because they do not contain u_t.

But instead we can consider the Burgers' equation itself: $F = u_{xx} + uu_x - u_t = 0$. In this case: $[F, \phi_2^1] = \mathcal{D}_x^2(\phi_2^1 - tF)$ and so the system is compatible and gives a (symmetric) solution of the equation.

5° The following examples constitute less trivial applications:

I. Let $\text{Hess}(u)$ be 2×2 matrix of the second derivatives $\left\| \frac{\partial^2 u}{\partial x^i \partial x^j} \right\|$.

(a) $\mathcal{E} = \{u_{xx} - u_{yy} = 0,\ \det \text{Hess}(u) = ae^{bu}\}$.

(b) $\mathcal{E} = \{u_{xx} + u_{yy} = 0,\ \det \text{Hess}(u) = ae^{bu}\}$.

II. $\mathcal{E} = \{u_{xx} + u_{yy} = (au + b)^{-1},\ \det \text{Hess}(u) = 0\}$.

Direct calculation shows that in each case $[F,G]|_{\mathcal{E}}=0$. So the system has (4-parametric) family of smooth solutions.

Remark 9. *Geometric explanation of this integrability: Both equations $\{F=0\}$ and $\{G=0\}$ in any of the above cases have group of the motions on the plane as the group of symmetries.*

The solutions can be found by Frobenius theorem. For example I(a) we have:

$$u=\varphi(x+y)+\psi(x-y),$$

where $\varphi=\Theta_{E_1,C_1,a/\lambda}$, $\psi=\Theta_{E_2,C_2,\lambda/4}$ are any of the following functions:

$$\Theta_{E,C,\sigma}(t)=\begin{cases} -\dfrac{1}{b}\ln\dfrac{\cosh(b\sqrt{2E}(t-C))\pm 1}{2bE\sigma^{-1}}, & \text{for } \ E>0,\\ -\dfrac{1}{b}\ln\dfrac{b\sigma(t-C)^2}{2}, & \text{for } \ E=0,\\ -\dfrac{1}{b}\ln\dfrac{\cos(b\sqrt{-2E}(t-C))+1}{2bE\sigma^{-1}}, & \text{for } \ E<0,\end{cases}$$

for $b\neq 0$ and $\Theta_{E,C,\sigma}(t)=\sigma t^2/2+Et+C$ for $b=0$.

With a little larger effort one solves I(b) and II.

3.2. Prolongation-projection scheme for second order PDEs

Usually the investigation of integrability for PDE $\mathcal{E}=\{F_\alpha=0\}$ (assume for simplicity the orders of F_α equal the same k) has two main ingredients: prolongation and projection. Actually we write $\mathcal{E}^{(1)}$ as the equation obtained by formal differentiations of the relations for $\mathcal{E}$: $\mathcal{E}^{(1)}=\{F_\alpha=0,\mathcal{D}_jF_\alpha=0\}$. Then we consider the projection $\tilde{\mathcal{E}}=\pi_{k+1,k}(\mathcal{E}^{(1)})$. If $\tilde{\mathcal{E}}\neq\mathcal{E}$ we restricts to this new equation and apply the machinery to it. Note that in light of the Weyl tensors theory the equation of $\tilde{\mathcal{E}}$ is $W_k(\mathcal{E})=0$. If $\tilde{\mathcal{E}}=\mathcal{E}$ we study $\mathcal{E}^{(1)}$ and so on (of course some regularity assumptions should be made on every stage, but we don't discuss them here).

Let's consider this process for a non-characteristic equation $\mathcal{E}\subset J^2(M^2)$ of codimension 2, $\mathcal{E}=\{F=G=0\}$. The obstruction for the existence of $\mathcal{E}^{(2)}$ is the function $[F,G]$, which due to diffeomorphism $\pi_{3,2}:\mathcal{E}^{(1)}\simeq\mathcal{E}$ can be considered as a function on J^2M. If $H=[F,G]$ vanishes on $\mathcal{E}$, then the equation $\mathcal{E}$ is (smooth) integrable.

Otherwise the equation $\tilde{\mathcal{E}}=\pi_{4,2}\mathcal{E}^{(2)}$ given by $\{F=G=H=0\}$ is strictly contained in $\mathcal{E}$. We suppose the equation $\tilde{\mathcal{E}}$ is regular of codimension 3, i.e. the symbol $g_{\tilde{\mathcal{E}}}=0$. This is equivalent to

$$J=\det\begin{pmatrix}F_{11} & F_{12} & F_{22}\\ G_{11} & G_{12} & G_{22}\\ H_{11} & H_{12} & H_{22}\end{pmatrix}\neq 0,$$

where as before $F_{ij}=\partial_{p_{ij}}(F)$ etc. Let us denote $F=I_1$, $G=I_2$, $H=I_3$.

Theorem 19. *For any regular equation* $\tilde{\mathcal{E}} = \{I_i = 0\}_{1\le i\le 3} \subset J^2M$ *of codimension 3 the only obstruction for (smooth local) integrability is the Weyl tensor*

$$W_2(\tilde{\mathcal{E}}) = J^{-1} \cdot dx^1 \wedge dx^2 \otimes e_g,$$

where

$$e_g = \sum_{\sigma=(i,j,k)\in S_3} \mathrm{sgn}(\sigma) \Big[\big((I_i)_{11}\mathcal{D}_1(I_j) + (I_i)_{12}\mathcal{D}_2(I_j)\big)(I_k)_{22}\partial_{p_1} \\ + \big((I_i)_{12}\mathcal{D}_1(I_j) + (I_i)_{22}\mathcal{D}_2(I_j)\big)(I_k)_{11}\partial_{p_2} \Big] \in \tau_x^*.$$

Proof. Actually in the case $J \neq 0$ the only non-trivial Spencer group is $H^{1,2}(\tilde{\mathcal{E}})$ and it is two-dimensional. The calculation of the Weyl tensor goes now by the usual scheme. □

Remark 10. *If* $\tilde{\mathcal{E}}$ *has the simplest form*

$$p_{11} = \varphi_{11}(x^i, u, p_i), \;\; p_{12} = \varphi_{12}(x^i, u, p_i), \;\; p_{22} = \varphi_{22}(x^i, u, p_i),$$

then the above integrability condition is simply

$$\mathcal{D}_2(\varphi_{11}) = \mathcal{D}_1(\varphi_{12}) \text{ and } \mathcal{D}_2(\varphi_{12}) = \mathcal{D}_1(\varphi_{22}).$$

Remark 11. *Another approach is to continue with Mayer brackets. Actually, if* $\tilde{\mathcal{E}}$ *is integrable, then both brackets* $[F, H] = 0$ *and* $[G, H] = 0$ *on* $\tilde{\mathcal{E}}$*. So if these functions are independent, they can be considered as components of* W_2*.*

So we see that the condition $H = 0$ on $\mathcal{E}$ is sufficient but not necessary for solvability (vs. integrability!) of $\mathcal{E}$.

3.3. Parallelizable webs on the plane

Consider the following PDE

$$F = p_{22} - 2p_{12} - f(x^i, u, p_i) = 0, \qquad G = p_{11} - 2p_{12} - g(x^i, u, p_i) = 0.$$

This equation occurs in the theory of 3-webs on the plane as an obstruction for the linearization of the web ([GMS]).

Let's calculate the Mayer bracket

$$\begin{aligned}[F, G] &= \mathcal{D}_1^2 F - 2\mathcal{D}_1\mathcal{D}_2 F - \mathcal{D}_2^2 G + 2\mathcal{D}_1\mathcal{D}_2 G \\ &= \mathcal{D}_2^2 g - \mathcal{D}_1^2 f + 2\mathcal{D}_1\mathcal{D}_2 f - 2\mathcal{D}_1\mathcal{D}_2 g.\end{aligned} \tag{12}$$

In what follows we consider the special case of parallelizable webs:

$$f = u \cdot (2p_1 - p_2), \qquad g = u \cdot (p_1 - 2p_2). \tag{13}$$

We calculate $\mathcal{D}_1 f = p_1(2p_1 - p_2) + u(2p_{11} - p_{12})$, $\mathcal{D}_2 g = p_2(p_1 - 2p_2) + u(p_{12} - 2p_{22})$ and so

$$\begin{aligned}\mathcal{D}_2^2 g - \mathcal{D}_1^2 f + 2\mathcal{D}_1\mathcal{D}_2 f - 2\mathcal{D}_1\mathcal{D}_2 g &= -u(2p_{111} - 3p_{112} - 3p_{122} + 2p_{222}) \\ &\quad - 3p_1(2p_{11} - 2p_{12} - p_{22}) + 3p_2(p_{11} + 2p_{12} - 2p_{22})\end{aligned}$$

Now the equation $\mathcal{E}^{(1)}$ is given by the relations $F = G = 0$ and

$$\begin{array}{ll}\mathcal{D}_1 F: & p_{122} - 2p_{112} + u(p_{12} - 2p_{11}) + p_1(p_2 - 2p_1) = 0, \\ \mathcal{D}_2 F: & p_{222} - 2p_{122} + u(p_{22} - 2p_{12}) + p_2(p_2 - 2p_1) = 0, \\ \mathcal{D}_1 G: & p_{111} - 2p_{112} + u(2p_{12} - p_{11}) + p_1(2p_2 - p_1) = 0, \\ \mathcal{D}_2 G: & p_{112} - 2p_{122} + u(2p_{22} - p_{12}) + p_2(2p_2 - p_1) = 0.\end{array}$$

Since $\mathcal{E}^{(1)} \simeq \mathcal{E}$ we can express p_{ijk} by the coordinates on $J^2 M$. In particular calculating $2\mathcal{D}_1 G - \mathcal{D}_2 G + 2\mathcal{D}_2 F - \mathcal{D}_1 F$ we get

$$2p_{111} - 3p_{112} - 3p_{122} + 2p_{222} = 0.$$

So we have:

$$\begin{aligned}[F, G] &= 3p_2(2p_{12} + u(p_1 - 2p_2) + 2p_{12} - 4p_{12} - 2u(2p_1 - p_2)) \\ &\quad - 3p_1(4p_{12} + 2u(p_1 - 2p_2) - 2p_{12} - 2p_{12} - u(2p_1 - p_2)) = 0.\end{aligned}$$

Let us call a point $x_2 \in \mathcal{E}$ *initial data of linearization* (pre-linearization) and call *linearization* any solution of $\mathcal{E}$. We proved

Proposition 20. *Consider a parallelizable 3-web on $\mathbb{R}^2$. Every initial data determines and uniquely the linearization of the web.* □

This statement was proved in [GMS] using a direct calculations with the Spencer complex via "standard and heavy computation". We think our approach is rather more economic.

The simple form of functions (13) is due to the fact that the curvature R of the Chern connection vanishes, $R = 0$, which is a criterion for the web to be parallelizable. If $R \neq 0$, then $[F, G]|_{\mathcal{E}} \neq 0$ and this is a new function H (calculated via (12)) defining a completely determined equation $\tilde{\mathcal{E}} \subset \mathcal{E}$.

Solvability conditions for this equation lead to description of the set $\mathcal{S} \subset \tilde{\mathcal{E}}$ on which initial data determines linearization (if determines then uniquely because $\tilde{\mathcal{E}}$ is overdetermined). Moreover this set is given by algebraic equations of high degrees and study of the projection $\pi_2 : \tilde{\mathcal{E}} \to M$ allows to conclude that the maximum number of linearizations for linearizable 3-web on complex plane $\mathbb{C}^2$ is 15 (of course one changes $\mathbb{R}$ to $\mathbb{C}$ when considering algebraic equations). This is the main result of the paper [GMS]. It can be also obtained by the method described in §3.2.

4. Historical remark

The *covariant* $[F, G]$ (which would be better called Jacobi-Lie-Mayer-Darboux bracket) for functions on J^1M is involved in the formulation of the criterion for transformation of the space $\mathbb{R}^{2n+1}(x_i, u, p_i)$ with *Pfaffian* system $du - \sum p_i dx_i$ to be contact ([K]). Namely the transformation $(x_i, u, p_i) \mapsto (X_i, U, P_i)$ is contact iff it satisfies the identities:

$$[X_i, X_j] = 0, \ [P_i, P_j] = 0, \ [X_i, U] = 0, \ p_i[P_j, X_j] = \delta_{ij}[P_i, U]. \tag{14}$$

This was first published by S. Lie in 1872/73 in *Verhandlungen der Akademie zu Christiania* (see [LE]) and then A. Mayer published an elementary approach in *Göttingger Nachrichten* (1873) and *Math. Annalen* (1874).

Also this bracket was studied by Jacobi (second Jacobi method).

In many classical textbooks the bracket (9) has no special name ([Gou], "square bracket" in [C]), in others it is referred to just as (generalized) Jacobi bracket ([Gu], [R]).

We however prefer to use Jacobi name for the bracket (10) since it satisfies the Jacobi identity:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0,$$

where $F, G, H \in C^\infty(J^\infty M)$, which means they are functions on some $J^k M$, and we define the bracket $\{,\}$ by the same formula (10).

The bracket $[,]$ of (9) satisfies a generalized Jacobi identity, which looks for $k = l = 1$ ([Li], [M]) as follows:

$$[F, [G, H]] + [G, [H, F]] + [H, [F, G]] = (\partial_u F)[G, H] + (\partial_u G)[H, F] + (\partial_u H)[F, G]$$

and is much uglier in the case $k > 1$.

Remark 12. *It is an interesting to study higher analogs of equations (14). Namely how to characterize* Lie transformations $\Phi : J^k\pi \to J^k\pi$ *in terms of the Mayer brackets? Of course Lie-Bäcklund theorem ([KLV]) describes the structure of Lie transformations, but it gives no answer to the above question.*

Let us also mention the following classical theorem:

Theorem 21. *Let $\mathcal{E}$ be an equation of the first order on a two-dimensional manifold given by relations $F(x^1, x^2, u, p_1, p_2) = 0, G(x^1, x^2, u, p_1, p_2) = 0$. Let the symbols of F and G be independent. Then $\mathcal{E}$ is integrable iff $[F, G]|_{\mathcal{E}} = 0$.*

Actually in one side (vanishing of $[F, G]$) the implication is contained in almost every textbook ([Gou], [C]), while to the other it seems to be implicit. It is a generalization of example in §1.5 for $n = 2$ and a particular case of our main theorem with $k = l = 1$. We write the proof for completeness.

Proof. The symbol of $\mathcal{E}$ is zero. So the Spencer sequence is

$$0 \to \Lambda^2\tau^* \to 0,$$

whence $H^{0,2}(\mathcal{E}) \simeq \mathbb{R}^1$ $(\tau \simeq \mathbb{R}^2)$ and $H^{i,2}(\mathcal{E}) = 0$ for $i > 0$. The only nontrivial Weyl tensor is $W_1 = \varpi^{-1}[F, G]$, where $\varpi = F'_{p_1}G'_{p_2} - F'_{p_2}G'_{p_1}$. □

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