Integrable systems and extensions of symmetry algebras

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Integrable nonlinear partial differential equations: applications

- hydrodynamics
- nonlinear optics
- cosmology
- relativity
- quantum field theory
- climate modelling
- chemistry
- biology
- finance
- ...

The key feature of integrable PDEs: Lax representations

(a.k.a. zero-curvature representations, Wahlquist–Estabrook prolongation structures, integrable extensions, inverse scattering transformation, differential coverings, ...):

- soliton solutions
- Bäcklund transformations
- nonlocal symmetries and nonlocal conservation laws
- recursion operators
- Darboux transformations
- ...

UNSOLVED PROBLEM: to find internal conditions ensuring existence of a Lax representation for a given PDE.

PDE $\mathcal{E}$:

$$F(x^i, u^\alpha, u^\alpha_{x^i}, u^\alpha_{x^i x^j}, \dots) = 0,$$

$i, j, \in \{1, \dots, n\}, \quad \alpha \in \{1, \dots, m\}$

Lax representation : an over-determined system

$$q_{a,x^k} = T_{a,k}(x^i, u^\alpha, u^\alpha_{x^i}, u^\alpha_{x^i x^j}, \dots q_b), \qquad a, b \in \mathbb{N},$$

compatibility conditions

$$(q_{a,x^i})_{x^j} = (q_{a,x^j})_{x^i} \quad \Longleftrightarrow \quad F = 0.$$

The Wahlquist–Estabrook form :

$$\tau_a = dq_a - \sum_{k=1}^{n} T_{a,k}(x^i, u^\alpha, u^\alpha_{x^i}, u^\alpha_{x^i x^j}, \dots q_b)\, dx^k$$

such that

$$d\tau_a = \sum_b \eta_{ab} \wedge \tau_b + \Omega_a, \qquad \Omega_a = 0 \quad \Longleftrightarrow \quad F = 0.$$

EXAMPLE. Liouville's equation:

$$u_{tx} = \mathrm{e}^u$$

Lax representation:

$$\left\{ \begin{array}{lcl} q_t & = & \frac{1}{2}\,u_t + \lambda\,\mathrm{e}^{\frac{1}{2}\,u+q}, \\ q_x & = & -\frac{1}{2}\,u_x - \frac{1}{2\lambda}\,\mathrm{e}^{\frac{1}{2}\,u-q}, \end{array} \right. \qquad \lambda \in \mathbb{R} \setminus \{0\} \qquad (1)$$

We have $(q_t)_x = (q_x)_t \quad \Longleftrightarrow \quad u_{tx} = \mathrm{e}^u$. Solve (1) for u_t, u_x:

$$\left\{ \begin{array}{lcl} u_t & = & 2\,q_t - 2\,\lambda\,\mathrm{e}^{\frac{1}{2}\,u+q}, \\ u_x & = & -2\,q_x - \frac{1}{\lambda}\,\mathrm{e}^{\frac{1}{2}\,u-q}, \end{array} \right. \qquad (2)$$

Then $(u_t)_x = (u_x)_t \quad \Longleftrightarrow \quad q_{tx} = 0$ (d'Alembert's equation). The Lax representation (1) defines a Bäcklund transformation between Liouville's equation and d'Alembert's equation.

The general solution to d'Alembert's equation:

$$q = F(t) + G(x),$$

substitute into (2) $\quad \Longrightarrow$

$$u = \ln\left(\frac{2\,\Phi'(t)\,\Psi'(x)}{(\Phi(t)+\Psi(x))^2}\right),$$

where

$$\left\{\begin{array}{lcl} \Phi'(t) & = & 2\,\lambda^2\,\mathrm{e}^{F(t)}, \\ \Psi'(x) & = & \mathrm{e}^{-G(x)}. \end{array}\right.$$

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EXAMPLE. The Lin–Reissner–Tsien equation:

$$u_{yy} = u_{tx} + u_x\, u_{xx}$$

The Lax representation:

$$\begin{cases} q_t & = & \dfrac{1}{3}\, q_x^3 - u_x\, q_x - u_y, \\ q_y & = & \dfrac{1}{2}\, q_x^2 - u_x. \end{cases}$$

- G.M. Kuz'mina, 1967,
- J. Gibbons, 1988; I.M. Krichever, 1988

Rename $q \mapsto q_0$, put $q_{0,x} = q_1$, ... $q_{k,x} = q_{k+1}$, ... $\Longrightarrow$

$$\left\{ \begin{array}{rcl} q_{k,t} & = & \left(\frac{1}{3}\, q_1^3 - u_x\, q_1 - u_y\right)_{\underbrace{x \dots x}_{k \text{ times}}}, \\ q_{k,x} & = & q_{k+1}, \\ q_{k,y} & = & \left(\frac{1}{2}\, q_1^2 - u_x\right)_{\underbrace{x \dots x}_{k \text{ times}}}. \end{array} \right.$$

$\diamond$

DEFINITION. A Lie algebra is a vector space $\mathfrak{g}$ endowed with a bilinear operation $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that there hold

1) $[x,y]=-[y,x]$ (skew symmetry),

2) $[[x,y],z]+[[z,x],y]+[[y,z],x]=0$ (Jacobi's identity).

EXAMPLE. $M\subseteq\mathbb{R}^m$: an open subset with coordinates $(x^1,\dots,x^m)$, the Lie algebra $\mathfrak{a}(M)$ of vector fields on M: linear differential operators

$$V=\sum_{k=1}^{m}a^k(x)\,\partial_{x^k}=\sum_k a^k\,\partial_{x^k}=a^k\,\partial_{x^k}$$

on $C^\infty(M)$,

$$[V,W]=V\circ W-W\circ V,$$

$$W=\sum_k b^k\partial_{x^k}\quad\Longrightarrow$$

$$[V,W]=\sum_{j,k}\left(a^j\,\partial_{x^j}(b^k)-b^j\,\partial_{x^j}(a^k)\right)\,\partial_{x^k}.$$

$\diamond$

EXAMPLE. Witt's algebra: $\mathfrak{w} \subsetneq \mathfrak{a}(\mathbb{R})$,

$$\mathfrak{w} = \langle v_k = t^k \, \partial_t \mid k \geq 0 \rangle, \qquad [v_i, v_j] = (j - i) \, v_{i+j-1}.$$

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DEFINITION. Let $\mathfrak{g}$ be a Lie algebra, let v_1, v_2, ... , v_n, ... be a basis of $\mathfrak{g}$. Then

$$[v_i, v_j] = \sum_k c_{ij}^k \, v_k, \qquad c_{ij}^k \in \mathbb{R}.$$

Constants c_{ij}^k are referred to as structure constants of $\mathfrak{g}$ in this basis.

REMARK: skew symmetry $\Longrightarrow$

$$c_{ij}^k = -c_{ji}^k,$$

Jacobi's identity $\Longrightarrow$

$$\sum_m \left(c_{ij}^m c_{mk}^n + c_{ki}^m c_{mj}^n + c_{jk}^m c_{mi}^n \right) = 0.$$

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DEFINITION: Let $\omega\colon \mathfrak{g} \to \mathbb{R}$ be a linear function (covector, 1-form). Define its differential $d\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ as

$$d\omega(v,w) = -\omega([v,w]).$$

For two 1-forms ω, θ define their exterior product $\omega \wedge \theta\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ as

$$\omega \wedge \theta(v,w) = \det \begin{pmatrix} \omega(v) & \theta(v) \\ \omega(w) & \theta(w) \end{pmatrix}.$$

For 1-forms ω^1, ..., ω^k define $\omega^1 \wedge \cdots \wedge \omega^k\colon \mathfrak{g} \times \cdots \times \mathfrak{g} \to \mathbb{R}$ as

$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, \ldots, v^k) = \det \begin{pmatrix} \omega^1(v_1) & \ldots & \omega^k(v_1) \\ \ldots & \ldots & \ldots \\ \omega^1(v_k) & \ldots & \omega^k(v_k) \end{pmatrix}.$$

Consider the dual forms ω^i for the basis v_j of the Lie algebra $\mathfrak{g}$ with the structure constants c^k_{ij}:

$$\omega^i(v_j) = \delta^i_j.$$

Then

$$d\omega^m(v_i, v_j) = -\omega^m([v_i, v_j]) = -\omega^m(c^k_{ij}\,\omega_k) = -c^k_{ij}\,\delta^m_k = -c^m_{ij},$$

$$\omega^p \wedge \omega^q(v_i, v_j) = \det \begin{pmatrix} \delta^p_i & \delta^q_i \\ \delta^p_j & \delta^q_j \end{pmatrix} = \begin{cases} 1, & p = i, \quad q = j, \\ -1, & p = j, \quad q = i, \\ 0, & \text{otherwise}, \end{cases}$$

$\Longrightarrow$

there hold the Maurer–Cartan structure equations of $\mathfrak{g}$:

$$d\omega^m = -\frac{1}{2}\sum_{p,q} c^m_{pq}\,\omega^p \wedge \omega^q$$

or, equivalently,

$$d\omega^m = -\sum_{p<q} c^m_{pq}\,\omega^p \wedge \omega^q.$$

EXAMPLE. For Witt's algebra $\mathfrak{w} = \langle v_k = \frac{1}{k!}\, t^k\, \partial_t \mid k \geq 0 \rangle$ take the dual forms ω^k such that $\omega^i(v_j) = \delta^i_j$. Consider the formal series

$$\Omega = \sum_{k \geq 0} \frac{h^k}{k!}\, \omega^k$$

with the formal parameter h such that $dh = 0$, define

$$\nabla_h(\Omega) = \partial_h\, \Omega = \sum_{k \geq 0} \frac{h^k}{k!}\, \omega^{k+1}.$$

Then the structure equations for $\mathfrak{w}$ can be written in the form

$$d\Omega = \nabla_h(\Omega) \wedge \Omega.$$

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For two 1-forms ω, θ define

$$d(\omega \wedge \theta) = d\omega \wedge \theta - \omega \wedge \theta.$$

Consider system of equations for 1-forms ω^1, ..., ω^k, ...

$$d\omega^m = -\frac{1}{2} \sum_{i,j} c^m_{ij} \, \omega^i \wedge \omega^j.$$

with some constants c^m_{ij}, w.l.o.g. $c^m_{ij} = -c^m_{ji}$.

THEOREM.

$$d(d\omega^k) = 0 \quad \Longleftrightarrow \quad \sum_m \left(c^m_{ij} c^n_{mk} + c^m_{ki} c^n_{mj} + c^m_{jk} c^n_{mi} \right) = 0.$$

Consider the Lie algebra $\mathfrak{a}(M) = \langle \sum_k a^k(x)\,\partial_{x^k} \mid a^k \in C^\infty(M) \rangle$ of vector fields on $M \subseteq \mathbb{R}^m$ with local coordinates $(x^1, \dots, x^m)$, denote by dx^1, ... , dx^m the 1-forms dual to ∂_{x^1}, ... , ∂_{x^m}, so

$$dx^i(\textstyle\sum_k a^k(x)\,\partial_{x^k}) = a^i(x).$$

Differential 1-forms on M are maps $\omega\colon \mathfrak{a}(M) \to C^\infty(M)$,

$$\omega = \textstyle\sum_i g_i(x)\,dx^i, \quad \omega(\sum_k a^k(x)\,\partial_{x^k}) = \sum_i g_i(x)\,a^i(x).$$

Differential n-forms on M, $n \le m$:

$$\omega = \sum_{i_1 < i_2 < \ldots < i_n} g_{i_1 i_2 \ldots i_n}(x)\,dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_n}.$$

Define exterior differential:

$$d(a(x)) = da(x) = \textstyle\sum_i \partial_{x^i} a(x)\,dx^i,$$

$$d(dx^j) = 0,$$

$$d(a(x)\,\theta) = da(x) \wedge \theta + a(x)\,d\theta.$$

THEOREM (Frobenius). Let ω^1, ... , ω^n be 1-forms on $M \subseteq \mathbb{R}^m$, $n \leq m$, such that

$$d\omega^i = \sum_{j=1}^{n} \pi^i_j \wedge \omega^j$$

for some 1-forms π^i_j on M. Then for each fixed point in M there exist local coordinates $(x^1, \dots, x^n, y^1, \dots, y^{m-n})$ on a neighborhood of this point such that in this neighborhood there holds

$$\omega^i = \sum_{j=1}^{n} g^i_j(x^1, \dots, x^n, y^1, \dots y^{m-n})\, dx^j, \quad i \in \{1, \dots, n\}.$$

EXAMPLE. The structure eqns of $\mathfrak{w}$: $d\Omega = \nabla_h(\Omega) \wedge \Omega$, that is,

$$d\omega^0 = \omega^1 \wedge \omega^0, \qquad (1)$$

$$d\omega^1 = \omega^2 \wedge \omega^0, \qquad (2)$$

$$d\omega^2 = \omega^3 \wedge \omega^0 - \omega^1 \wedge \omega^2, \qquad (3)$$

$$d\omega^3 = \omega^4 \wedge \omega^0 - 2\,\omega^1 \wedge \omega^3, \qquad (4)$$

$$\dots \quad \dots \quad \dots$$

Apply Frobenius' theorem to (1) $\Longrightarrow$ $\omega^0 = a_0\,dt$, then

$$da_0 \wedge dt = \omega^1 \wedge a_0\,dt \quad \Longrightarrow \quad \omega^1 = \frac{da_0}{a_0} + a_1\,dt,$$

substitute into (2) :

$$da_1 \wedge dt = \omega^2 \wedge a_0\,dt \quad \Longrightarrow \quad \omega^2 = \frac{da_1}{a_0} + a_2\,dt,$$

substitute into (3) :

$$da_2 \wedge dt = \left(a_0\omega^3 + \frac{a_1 da_1}{a_0} - \frac{a_2 da_0}{a_0}\right) \wedge dt \quad \Longrightarrow$$

$$\omega^3 = \frac{da_2}{a_0} - \frac{a_1}{a_0^2}\,da_1 + \frac{a_2}{a_0^2}\,da_0 + a_3\,dt,$$

etc. $\diamond$

REMARK. Take 1-form $\omega = a\,dt$ on $\mathbb{R}_t \times \mathbb{R}_a$. Consider a local diffeomorphism $\Phi\colon \mathbb{R}_t \times \mathbb{R}_a \to \mathbb{R}_t \times \mathbb{R}_a$ such that

$$\Phi^* \omega = \omega,$$

that is, $\Phi\colon (t,a) \mapsto (\tilde{t}, \tilde{a}) = (\phi(t,a), \psi(t,a))$, and

$$\tilde{a}\,d\tilde{t} = \psi(t,a)\,d\phi(t,a) = a\,dt.$$

Then $\quad \partial_a \phi = 0 \quad \Longrightarrow \quad \tilde{t} = \phi(t) \quad \Longrightarrow \quad$ Φ is projectable on $\mathbb{R}_t$, and the projection is a local diffeomorphism.

Conversely, let $\phi\colon \mathbb{R}_t \to \mathbb{R}_t$ be a local diffeomorphism. Define the lift $\Phi\colon \mathbb{R}_t \times \mathbb{R}_a \to \mathbb{R}_t \times \mathbb{R}_a$ by

$$\Phi\colon (t,a) \mapsto (t, a\,(\phi'(t))^{-1}).$$

Then $\Phi^* \omega = \omega$.

E.g., $\phi_k(t) = t + \epsilon\, t^k + \ldots$ has infinitesimal generator $v_k = t^k\,\partial_t$.

Note that $d\omega = \pi \wedge \omega$ for $\pi \equiv da/a \bmod \omega$, so $\mathfrak{w}$ and equation $d\omega = \pi \wedge \omega$ are related. $\diamond$

REMARK. Consider $d\omega^0 = \omega^1 \wedge \omega^0$, then

$$0 = d(d\omega^0) = d\omega^1 \wedge \omega^0 - \omega^1 \wedge d\omega^0 = d\omega^1 \wedge \omega^0,$$

$\Longrightarrow$

$$d\omega^1 = \omega^2 \wedge \omega^0$$

for some ω^2. Further,

$$0 = d(d\omega^1) = d\omega^2 \wedge \omega^0 - \omega^2 \wedge d\omega^0 = (d\omega^2 - \omega^2 \wedge \omega^1) \wedge \omega^0,$$

$\Longrightarrow$

$$d\omega^2 = \omega^2 \wedge \omega^1 + \omega^3 \wedge \omega^0,$$

then

$$d(d\omega^2) = 0 \quad \Longrightarrow \quad d\omega^3 = \omega^4 \wedge \omega^0 - 2\,\omega^1 \wedge d\omega^3,$$

etc., the whole system of the structure equations for $\mathfrak{w}$ is recoverable from the first equation.

$\diamond$

DEFINITION. Let $\mathfrak{g}$ be a Lie algebra with the structure eqns

$$d\omega^i = \sum_{j<k} c^i_{jk}\,\omega^j \wedge \omega^k.$$

A 2-form $\Omega = \sum_{j<k} a^i_{jk}\,\omega^j \wedge \omega^k$ is a 2-cocycle on $\mathfrak{g}$ if $d\Omega = 0$.

Denote the linear space of 2-cocycles by $Z^2(\mathfrak{g})$.

DEFINITION. 2-cocycle of the form

$$\sum_i b_i\, d\omega^i = \sum_i \sum_{j<k} b_i\, c^i_{jk}\,\omega^j \wedge \omega^k$$

is referred to as a 2-coboundary. Denote the linear space of 2-coboundaries by $B^2(\mathfrak{g})$.

DEFINITION. The second cohomology group of $\mathfrak{g}$:

$$H^2(\mathfrak{g}) = Z^2(\mathfrak{g})/B^2(\mathfrak{g}).$$

Let $\Omega = \sum_{j<k} a_{jk}\,\omega^j \wedge \omega^k \in H^2(\mathfrak{g})$, then we can extend $\mathfrak{g}$: consider the Lie algebra $\hat{\mathfrak{g}}$ with the structure equations

$$\left\{ \begin{array}{lcl} d\omega^i & = & \sum_{j<k} c^i_{jk}\,\omega^j \wedge \omega^k, \\ d\theta & = & \sum_{j<k} a_{jk}\,\omega^j \wedge \omega^k, \end{array} \right. \tag{1}$$

or in terms of the dual vectors v_1, ... , v_k, ... , w to ω^1, ... , ω^k, ... , θ:

$$\left\{ \begin{array}{lcl} [v_j, v_k] & = & -a_{jk}\,w - \sum_i c^i_{jk}\,v_i, \\ [v_j, w] & = & 0. \end{array} \right.$$

Note that w is in the center $C(\mathfrak{g}) = \{ x \in \mathfrak{g} \mid \forall y \in \mathfrak{g}\,([x,y] = 0)\,\}$ of $\mathfrak{g}$. Therefore $\hat{\mathfrak{g}}$ is called the central extension of $\mathfrak{g}$ generated by the 2-cocycle Ω.

$\diamond$

Suppose there is 1-form α on $\mathfrak{g}$ such that $d\alpha = 0$, so the structure equations are of the form

$$\left\{ \begin{array}{lcl} d\alpha & = & 0, \\ d\omega^i & = & \sum\limits_{j<k} c^i_{jk}\,\omega^j \wedge \omega^k + \sum\limits_j \alpha \wedge b^i_j \omega^j. \end{array} \right.$$

For fixed $\lambda \in \mathbb{R}$ define twisted differential by

$$d_{\lambda\alpha}\theta = d\theta - \lambda\,\alpha \wedge \theta.$$

Then $\quad d\alpha = 0 \quad$ implies $\quad d^2_{\lambda\alpha} = 0.$

DEFINITION. A 2-form Ω on $\mathfrak{g}$ is a twisted 2-cocycle if $d_{\lambda\alpha}\Omega = 0$. The linear space of twisted 2-cocycles is denoted by $Z^2_{\lambda\alpha}(\mathfrak{g})$. $\Omega \in Z^2_{\lambda\alpha}(\mathfrak{g})$ is a twisted coboundary if $\Omega = d_{\lambda\alpha}\omega$ for some 1-form ω on $\mathfrak{g}$. The linear space of twisted coboundaries is denoted by $B^2_{\lambda\alpha}(\mathfrak{g})$. The second twisted cohomology group of $\mathfrak{g}$ is defined as the factor space $H^2_{\lambda\alpha}(\mathfrak{g}) = Z^2_{\lambda\alpha}(\mathfrak{g})/B^2_{\lambda\alpha}(\mathfrak{g})$.

Suppose $\Omega \in H^2_{\lambda\alpha}(\mathfrak{g})$ and define the extension $\hat{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ by the structure equations

$$\left\{ \begin{array}{lcl} d\alpha & = & 0, \\ d\omega^i & = & \sum\limits_{j<k} c^i_{jk}\, \omega^j \wedge \omega^k + \sum\limits_{j} \alpha \wedge b^i_j \omega^j, \\ d\theta & = & \lambda\, \alpha \wedge \theta + \Omega. \end{array} \right.$$

To verify that the above structure equations define a Lie algebra we have to check condition $d(d\theta) = 0$:

$$\begin{aligned} d(d\theta) &= d(\lambda\, \alpha \wedge \theta + \Omega) = \lambda\, d\alpha \wedge \theta - \lambda\, \alpha \wedge d\theta + d\Omega \\ &= -\lambda\, \alpha \wedge (\lambda\, \alpha \wedge \theta + \Omega) + d\Omega = -\lambda\, \alpha \wedge \Omega + d\Omega = 0. \end{aligned}$$

The Lie algebra $\hat{\mathfrak{g}}$ is called the extension of $\mathfrak{g}$ generated by the twisted 2-cocycle Ω.

DEFINITION. A derivation of a Lie algebra $\mathfrak{g}$ is a linear map $D\colon \mathfrak{g} \to \mathfrak{g}$ such that

$$D([v,w]) = [D(v),w] + [v,D(w)].$$

The linear space of all derivations: $\mathrm{Der}(\mathfrak{g})$

REMARK. $D_1, D_2 \in \mathrm{Der}(\mathfrak{g}) \implies D_1 \circ D_2 - D_2 \circ D_1 \in \mathrm{Der}(\mathfrak{g})$, so $\mathrm{Der}(\mathfrak{g})$ is a Lie algebra with $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

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DEFINITION. For $v \in \mathfrak{g}$ define $\mathrm{ad}_v\colon \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}_v\colon w \mapsto [v,w]$. Then $ad_v \in \mathrm{Der}(\mathfrak{g})$. Put $\mathrm{Der}_{\mathrm{inn}}(\mathfrak{g}) = \{ad_v \in \mathrm{Der}(\mathfrak{g}) \mid v \in \mathfrak{g}\}$. Denote $\mathrm{Der}_{\mathrm{out}}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})/\mathrm{Der}_{\mathrm{inn}}(\mathfrak{g})$.

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EXAMPLE. Let $D \in \mathrm{Der}_{\mathrm{out}}(\mathfrak{g})$. Extend $\mathfrak{g}$: put $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}\, w$ (sum of vector spaces) and define $[w, v] = D(v)$ for each $v \in \mathfrak{g}$. In other words, if v_i is a basis of $\mathfrak{g}$, then

$$D(v_i) = \sum_j b_i^j\, v_j \quad \Longrightarrow \quad [w, v_i] = \sum_j b_i^j\, v_j.$$

If ω^i, β are dual forms for v_i, w, then

$$\left\{ \begin{array}{lcl} d\beta & = & 0, \\ d\omega^i & = & \displaystyle\sum_{j<k} c_{jk}^i\, \omega^j \wedge \omega^k - \beta \wedge \sum_j b_j^i\, \omega^j. \end{array} \right.$$

Likewise, if $D_s(v_i) = \displaystyle\sum_j b_{si}^j\, v_j$ and $[D_p, D_q] = \displaystyle\sum_r a_{pq}^r\, D_r$ for $D_1, \ldots, D_s, \cdots \in \mathrm{Der}_{\mathrm{out}}(\mathfrak{g})$, then define $\hat{\mathfrak{g}}$ by

$$\left\{ \begin{array}{lcl} d\beta^r & = & \displaystyle -\sum_{p<q} a_{pq}^r\, \beta^p \wedge \beta^q, \\ d\omega^i & = & \displaystyle\sum_{j<k} c_{jk}^i\, \omega^j \wedge \omega^k - \sum_{s,j} b_{sj}^i\, \beta^s \wedge \omega^j. \end{array} \right.$$

$\diamond$

EXAMPLE. Let $\mathfrak{g}$ be a $\mathbb{Z}_+$–graded Lie algebra, that is, $\mathfrak{g} = \sum_{q=0}^{\infty} \mathfrak{g}_q$ (sum of vector spaces) and $[\mathfrak{g}_p, \mathfrak{g}_q] \subseteq \mathfrak{g}_{p+q}$. Then $D\colon \mathfrak{g} \to \mathfrak{g}$, $D\colon v \mapsto p\, v$ for $v \in \mathfrak{g}_p$, is a derivation of $\mathfrak{g}$. Let v_1, ... , v_n, ... be a graded basis, that is, $v_i \in \mathfrak{g}_{p(i)}$ for some $p(i)$. Then D defines an extension of $\mathfrak{g}$ with the structure equations

$$\begin{cases} d\alpha &= 0, \\ d\omega^i &= p(i)\, \alpha \wedge \omega^i + \sum_{j<k} c^i_{jk}\, \omega^j \wedge \omega^k. \end{cases}$$

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EXAMPLE. The (commutative associative unital) algebra of truncated polynomials of degree N: $\mathbb{R}_N[s] = \mathbb{R}[s]/(s^{N+1} = 0)$. Let $\mathfrak{g}$ be a Lie algebra with a basis $\{v_i\}$ and structure constants c^i_{jk}. The current algebra: $\mathbb{R}_N[s] \otimes \mathfrak{g} = \langle s^k \otimes v_i \mid k \in \{0, \dots, N\} \rangle$,

$$[s^k \otimes v_i, s^m \otimes v_j] = \begin{cases} s^{k+m} \otimes [v_i, v_j], & k+m \le N, \\ 0, & k+m > N. \end{cases}$$

The dual forms for $s^k \otimes v_i$: ω^j_m such that $\omega^j_m(s^k \otimes v_i) = \delta^k_m\,\delta^j_i$. Then the structure equations for $\mathbb{R}_N[s] \otimes \mathfrak{g}$ are

$$d\omega^i_m = -\sum_{p+q=m}\sum_{j<k} c^i_{jk}\,\omega^j_p \wedge \omega^k_q.$$

The grading: $[s^k \otimes \mathfrak{g}, s^m \otimes \mathfrak{g}] \subseteq s^{k+m} \otimes \mathfrak{g} \quad \Longrightarrow \quad$ the extension

$$\left\{ \begin{array}{lcl} d\alpha & = & 0, \\ d\omega^i_m & = & m\,\alpha \wedge \omega^i - \displaystyle\sum_{p+q=m}\sum_{j<k} c^i_{jk}\,\omega^j_p \wedge \omega^k_q. \end{array} \right.$$

$\diamond$

EXAMPLE: Consider

$$\mathbb{R}_N[s] \otimes \mathfrak{w} = \langle v^m_k = \tfrac{1}{k!}\,s^m\,t^k \partial_t \mid m \in \{0, \dots, N\}, k \geq 0 \rangle.$$

For the dual forms ω^k_m put

$$\Omega = \sum_{j=0}^{N}\sum_{k=0}^{\infty} \frac{1}{k!}\,s^j h^k\,\omega^k_j,$$

Then the structure equations of $\mathbb{R}_N[s] \otimes \mathfrak{w}$ are $d\Omega = \nabla_h(\Omega) \wedge \Omega$.

The grading: $[s^p \otimes \mathfrak{w}, s^q \otimes \mathfrak{w}] \subseteq s^{p+q} \otimes \mathfrak{w} \implies$ extension with the structure equations

$$\left\{ \begin{array}{lcl} d\alpha & = & 0, \\ d\Omega & = & s\,\alpha \wedge \nabla_s(\Omega) + \nabla_h(\Omega) \wedge \Omega \end{array} \right.$$

◇

PDE:

$$\mathcal{E} = \{F_r(x^i, u^\alpha, u^\alpha_{x^i}, u^\alpha_{x^i x^j}, \dots) = 0\},$$

(An infinitesimal generator of) a symmetry of $\mathcal{E}$:

$$\phi = (\phi^1(x^i, u^\beta, u^\beta_{x^i}, \dots), \dots, \phi^m(x^i, u^\beta, u^\beta_{x^i}, \dots))$$

such that

$$F_r(x^i, u^\alpha + \epsilon\,\phi^\alpha, (u^\alpha + \epsilon\,\phi^\alpha)_{x^i}, (u^\alpha + \epsilon\,\phi^\alpha)_{x^i x^j}, \dots) = o(\epsilon).$$

In other words,

$$\mathbf{E}_\phi(F) = \sum_{i_1,\dots,i_n,\alpha} \frac{\partial F_r}{\partial u^\alpha_{i_1,\dots,i_n}}\, D^{i_1}_{x^1} \circ \cdots \circ D^{i_n}_{x^n}(\phi^\alpha)|_{F=0} = 0,$$

where the total derivatives are

$$D_i = \partial_{x^i} + \sum_{\alpha,I} u^\alpha_{I+1_i}\,\partial_{u^\alpha_I}.$$

The linear space of symmetries is a Lie algebra $\mathrm{Sym}(\mathcal{E})$ with

$$[\phi, \psi] = \mathbf{E}_\phi(\psi) - \mathbf{E}_\psi(\phi).$$

EXAMPLE. The hyper-CR equation for Einstein–Weyl structures:

$$u_{yy} = u_{tx} + u_y\, u_{xx} - u_x\, u_{xy} \qquad (\mathcal{E}_1)$$

- G.M. Kuz′mina, 1967
- V.G. Mikhalev, 1992
- M.V. Pavlov, 2003
- M. Dunajski, 2004

Lax representation with a non-removable parameter:

$$\left\{ \begin{array}{lcl} v_t & = & (\lambda^2 - \lambda\, u_x - u_y)\, v_x, \\ v_y & = & (\lambda - u_x)\, v_x. \end{array} \right.$$

Algebra of contact symmetries $\mathrm{Sym}(\mathcal{E}_1) = \mathfrak{p}_\diamond \ltimes \mathfrak{p}_{4,\infty}$ with

$$\mathfrak{p}_\diamond = \langle \psi_0, \psi_1 \rangle, \qquad \mathfrak{p}_{4,\infty} = \langle \phi_0(A), \dots \phi_3(A) \rangle = \mathbb{R}_3[h] \otimes \mathfrak{w},$$

where

$$\begin{aligned}
\phi_0(A) &= -A u_t - \tfrac{1}{2}\, y\, (y u_x - 2x) A'' - A'(x u_x + y u_y - u) + \tfrac{1}{6}\, y^3\, A''', \\
\phi_1(A) &= -y\, A'\, u_x - A\, u_y + x\, A' + \tfrac{1}{2}\, y^2\, A'', \\
\phi_2(A) &= -A\, u_x + y\, A', \\
\phi_3(A) &= A, \\
\psi_0 &= -2\, x\, u_x - y\, u_y + 3\, u, \\
\psi_1 &= -y\, u_x + 2\, x,
\end{aligned}$$

and $A = A(t)$ are arbitrary functions.

Commutators:

$$\begin{aligned}
\{\phi_i(A), \phi_j(B)\} &= \phi_{i+j}(A\, B' - B\, A'), \\
\{\psi_i, \phi_k(A)\} &= -k\, \phi_{k+i}(A), \\
\{\psi_0, \psi_1\} &= -\psi_1.
\end{aligned}$$

Maurer–Cartan forms: α_0, α_1, $\theta_{k,n}$, $k \in \{0, ..., 3\}$, $n \geq 0$,

$$\alpha_i(\psi_j) = \delta_{ij}, \qquad \alpha_i(\phi_k(t^n)) = 0,$$

$$\theta_{k,n}(\psi_i) = 0, \qquad \tfrac{1}{m!}\,\theta_{k,n}(\phi_l(t^m)) = \delta_{kl}\delta_{nm}$$

Denote $\quad \Theta = \displaystyle\sum_{k=0}^{3}\sum_{m=0}^{\infty} \frac{1}{m!}\, h_0^k\, h_1^m\, \theta_{k,m},$

then the structure equations of $\mathrm{Sym}(\mathcal{E}_1)$ take the form

$$\left\{\begin{array}{lcl} d\alpha_0 & = & 0, \\ d\alpha_1 & = & \alpha_0 \wedge \alpha_1, \\ d\Theta & = & \nabla_{h_1}(\Theta) \wedge \Theta + (h_0\,\alpha_0 + h_0^2\,\alpha_1) \wedge \nabla_{h_0}(\Theta), \end{array}\right.$$

(recall that $h_0^k = 0$ when $k > 3$).

From the structure equations we have:

$$H^1(\mathrm{Sym}(\mathcal{E}_1)) = \mathbb{R}\,\alpha_0,$$

$$H^2_{c\alpha_0}(\mathfrak{p}_\diamond) = \begin{cases} \mathbb{R}\,[\alpha_0 \wedge \alpha_1], & c = 1, \\ \{[0]\}, & c \neq 1. \end{cases}$$

Moreover,

$$H^2_{\alpha_0}(\mathfrak{p}_\diamond) \subseteq H^2_{\alpha_0}(\mathrm{Sym}(\mathcal{E}_1)),$$

hence equation

$$d\sigma = \alpha_0 \wedge \sigma + \alpha_0 \wedge \alpha_1$$

is compatible with the structure equations of $\mathrm{Sym}(\mathcal{E}_1)$ and defines a non-central extension of this Lie algebra.

We have ($r \neq 0$, $q, s \in \mathbb{R}$ are parameters):

$$\alpha_0 = dq,$$

$$\alpha_1 = -\mathrm{e}^q \, ds,$$

$$\sigma = \mathrm{e}^q \, (dv - q \, ds),$$

$$\theta_{0,0} = r \, dt,$$

$$\theta_{1,0} = r \, \mathrm{e}^q \, (dy + p_1 \, dt),$$

$$\theta_{2,0} = r \, \mathrm{e}^{2q} \, (dx + (p_1 - s) \, dy + p_2 \, dt),$$

$$\theta_{3,0} = p \, \mathrm{e}^{3q} \, (du + (p_1 - 2\, s) \, dx + (p_2 - s \, p_1 + s^2) \, dy + p_3 \, dt).$$

We know

$$\theta_{3,0} = p \, \mathrm{e}^{3q} \, (du - u_x \, dx - u_y \, dy - u_t \, dt)$$

$$\Rightarrow \qquad \left\{ \begin{array}{lcl} p_1 - 2\, s & = & -u_x, \\ p_2 - s \, p_1 + s^2 & = & -u_y, \\ p_3 & = & -u_t. \end{array} \right.$$

Consider

$$\sigma - \theta_{2,0} =$$
$$\mathrm{e}^{q} \, \left(dv - q \, ds - r \, \mathrm{e}^{q} \, \left(dx + (s - u_x) \, dy + (s^2 - s \, u_x - u_y) \, dt \right) \right),$$

rename $\quad q = v_s, \quad r = v_x \, \exp(-v_s) \qquad \Rightarrow$

$$\sigma - \theta_{2,0} =$$
$$\mathrm{e}^{v_s} \, \left(dv - v_s \, ds - v_x \, (dx + (s - u_x) \, dy + (s^2 - s \, u_x - u_y) \, dt) \right).$$

Then $\sigma - \theta_{2,0} = 0 \qquad \Leftrightarrow$

$$\left\{ \begin{array}{lcl} v_t & = & (s^2 - s \, u_x - u_y) \, v_x, \\ v_y & = & (s - u_x) \, v_x. \end{array} \right.$$

$\Longrightarrow$

$\sigma - \theta_{2,0}$ is the Wahlquist–Estabrook form for the above Lax representation of the hyper-CR equation, $\lambda = s$.

Another linear combination: $\sigma - \theta_{0,0} - \theta_{1,0} - \theta_{2,0} = 0 \quad \Leftrightarrow$

$$\left\{ \begin{array}{rcl} v_t & = & \left(s^2 - s\,u_x - u_y + \mathrm{e}^{-v_s}\,(2\,s - u_x) + \mathrm{e}^{-2\,v_s}\right)\,v_x, \\ v_y & = & \left(s - u_x + \mathrm{e}^{-v_s}\right)\,v_x. \end{array} \right.$$

$\Longrightarrow$

$\sigma - \theta_{0,0} - \theta_{1,0} - \theta_{2,0}$ is the Wahlquist–Estabrook form for the new Lax representation of the hyper-CR equation .

EXAMPLE: The Lin–Reissner–Tsien equation

$$u_{yy} = u_{tx} + u_x\, u_{xx}. \qquad (\mathcal{E}_2)$$

The structure equations for $\mathrm{Sym}(\mathcal{E}_2)$:

$$\begin{cases} d\alpha &= 0, \\ d\Theta &= s\,\alpha \wedge \nabla_s(\Theta) + \nabla_h(\Theta) \wedge \left(\Theta - \dfrac{1}{3}\, s\, \nabla_s(\Theta)\right), \end{cases}$$

where

$$\Theta = \sum_{k=0}^{4} \sum_{m=0}^{\infty} \frac{1}{m!}\, s^k\, h^m\, \theta_{km},$$

with $\theta_{3,0} = 0$ and $s^k = 0$ for $k > 4$. The Maurer-Cartan forms:

$$\alpha = dq/q, \quad \theta_{0,0} = r\, dt, \quad \theta_{1,0} = q\, r^{2/3}\, (dy + a_1\, dt),$$

$$\theta_{2,0} = q^2\, r^{1/3}\, (dx + \tfrac{2}{3}\, a_1\, dy + a_2\, dt),$$

$$\theta_{4,0} = q^4\, r^{-1/3}\, (du - u_t dt - u_x dx - u_y dy).$$

THEOREM.

$$H^2_{\lambda\alpha}(\mathrm{Sym}(\mathcal{E}_2)) = \begin{cases} \langle[\Omega]\rangle, & \lambda = 3, \\ \{[0]\}, & \lambda \neq 3, \end{cases}$$

where

$$\Omega = \theta_{3,1} \wedge \theta_{0,0} + \tfrac{2}{3}\,\theta_{2,1} \wedge \theta_{1,0} + \tfrac{1}{3}\,\theta_{1,1} \wedge \theta_{2,0}.$$

COROLLARY. Equation

$$d\tau = 3\,\alpha \wedge \tau + \theta_{3,1} \wedge \theta_{0,0} + \tfrac{2}{3}\,\theta_{2,1} \wedge \theta_{1,0} + \tfrac{1}{3}\,\theta_{1,1} \wedge \theta_{2,0}$$

is compatible with the structure equations of $\mathrm{Sym}(\mathcal{E}_2)$.

Integrate $\Longrightarrow$ the Wahlquist–Estabrook form

$$\tau = q^3\,\left(dv - v_x\,dx - \left(\frac{1}{3}\,v_x^3 - u_x\,v_x - u_y\right)\,dt - \left(\frac{1}{2}\,v_x^2 - u_x\right)\,dy\right).$$

for the Lax representation

$$\begin{cases} v_t = \dfrac{1}{3}\,v_x^3 - u_x\,v_x - u_y, \\ v_y = \dfrac{1}{2}\,v_x^2 - u_x. \end{cases}$$

◇

EXAMPLE. Consider $\mathbb{R}_N \otimes \mathfrak{w}$ as the vector space of functions $f(s,t) = f_0(t) + \cdots + f_N(t)\, s^N$ with $[f,g] = f\, g_t - g\, f_t$ and $s^k = 0$ for $k > N$. For fixed $\epsilon \in \mathbb{R}$ define the deformation $\mathfrak{g}(N,\epsilon)$ of $\mathbb{R}_N \otimes \mathfrak{w}$ as the same vector space with new bracket

$$[f,g]_\epsilon = f\, g_t - g\, f_t + \epsilon\, s\, (f_s\, g_t - g_s\, f_t).$$

Let θ_{km} be the dual forms to $\{s^k t^m \,|\, 0 \le k \le N, m \ge 0\}$, put

$$\Theta = \sum_{k=0}^{N} \sum_{m \ge 0} \frac{1}{m!} s^k\, h^m\, \theta_{k,m}.$$

Then the structure equations for $\mathfrak{g}(N,\epsilon)$ are

$$d\Theta = \nabla_h(\Theta) \wedge (\Theta + \epsilon\, s\, \nabla_s(\Theta)).$$

The deformation preserves the grading $s^k \otimes \mathfrak{w} \quad \Longrightarrow \quad$ there is the extension $\hat{g}(N,\epsilon)$ of $\mathfrak{g}(N\epsilon)$ with the structure equations

$$\left\{ \begin{array}{lcl} d\alpha & = & 0, \\ d\Theta & = & s\, \alpha \wedge \nabla_s(\Theta) + \nabla_h(\Theta) \wedge (\Theta + s\, \nabla_s(\Theta)). \end{array} \right.$$

THEOREM.

$$H^2_{\lambda\alpha}(\hat{g}(N,\epsilon)) = \begin{cases} \langle [\Phi_r] \rangle, & \epsilon = -\dfrac{2}{r},\ r \in \{2,\dots,N\},\ \lambda = r, \\ \{[0]\}, & \text{otherwise}, \end{cases}$$

where $\quad \Phi_r = \sum_{m=0}^{[r/2]} (r - 2\,m)\,\theta_{r-m,0} \wedge \theta_{m,0}.$

E.g., $N = r = 3 \quad \Longrightarrow \quad \epsilon = -\frac{2}{3},$ therefore equation

$$d\tau = 3\,\alpha \wedge \tau + 3\,\theta_{3,0} \wedge \theta_{0,0} + \theta_{2,0} \wedge \theta_{1,0}$$

is compatible with the structure equations of $\hat{\mathfrak{g}}(3, -2/3)$, integrate $\Longrightarrow$ the Wahlquist–Estabrook form of the Lax representation

$$\begin{cases} v_t &= u - (u_x^2 + u_y)\,v_x, \\ v_y &= x - u_x\,v_x \end{cases}$$

of equation

$$u_{yy} = u_{tx} + (u_y - u_x^2)\,u_{xx} - 3\,u_x\,u_{xy}. \qquad \diamond$$

EXAMPLE. Integrable hierarchy associated to $\mathcal{E}_1$.

Consider the Lie algebra

$$\mathfrak{p}_\diamond \ltimes \mathfrak{p}_{n+1,\infty}, \text{ where } \mathfrak{p}_{n+1,\infty} = \mathbb{R}_n[h] \otimes \mathfrak{w}.$$

The structure equations:

$$\begin{cases} d\alpha_0 &= 0, \\ d\alpha_1 &= \alpha_0 \wedge \alpha_1, \\ d\Theta &= \nabla_{h_1}(\Theta) \wedge \Theta + (h_0\, \alpha_0 + h_0^2\, \alpha_1) \wedge \nabla_{h_0}(\Theta) \end{cases}$$

with

$$\Theta = \sum_{k=0}^{n} \sum_{m=0}^{\infty} \frac{1}{m!}\, h_0^k\, h_1^m\, \theta_{k,m}.$$

Rename: $t \mapsto t_{n-1}$, $y \mapsto t_{n-2}$, $x \mapsto t_{n-3}$, then

$$\theta_{n,0} = r\,\mathrm{e}^{nq} \left(du - \sum_{i=0}^{n-1} u_{t_i} dt_i\right)$$

$\Rightarrow$

$$\sigma - \theta_{n-1,0} =$$
$$\mathrm{e}^{v_s} \left(dv - v_s\,ds - v_{t_0}\,dt_0 - \sum_{i=1}^{n-1} \left(s^i - \sum_{j=0}^{i-1} s^{i-j-1}\,u_{t_j}\right) v_{t_0}\,dt_i\right).$$

$$\sigma - \theta_{n-1,0} = 0 \quad \Longrightarrow$$

$$\left\{ \begin{array}{lcl} v_{t_1} & = & (s - u_{t_0})\, v_{t_0}, \\ v_{t_2} & = & (s^2 - s\, u_{t_0} - u_{t_1})\, v_{t_0}, \\ & & \dots \\ v_{t_i} & = & \left(s^i - \sum\limits_{j=0}^{i-1} s^{i-j-1}\, u_{t_j} \right) v_{t_0}, \\ & & \dots \\ v_{t_{n-1}} & = & \left(s^{n-1} - s^{n-2}\, u_{t_0} - s^{n-3}\, u_{t_1} - \dots - s\, u_{t_{n-3}} - u_{t_{n-2}}\right)\, v_{t_0}. \end{array} \right.$$

- M. Dunajski, 2004,
- M.V. Pavlov, 2003; L.V. Bogdanov, M.V. Pavlov, 2017

Denote by $\mathcal{H}_{n-1}$ the compatibility conditions of this system. Then $\mathcal{H}_2$ is the hyper-CR equation

$$u_{t_1t_1} = u_{t_0t_2} + u_{t_1}\,u_{t_0t_0} - u_{t_0}\,u_{t_0t_1},$$

$\mathcal{H}_3$ is the hyper-CR equation plus

$$u_{t_1t_2} = u_{t_0t_3} + u_{t_2}\,u_{t_0t_0} - u_{t_0}\,u_{t_0t_2},$$
$$u_{t_1t_3} = u_{t_2t_2} + u_{t_1}\,u_{t_0t_2} - u_{t_2}\,u_{t_0t_1},$$

$\mathcal{H}_4$ consists of $\mathcal{H}_3$ plus

$$u_{t_0t_4} = u_{t_2t_2} + u_{t_0}\,u_{t_0t_3} - u_{t_3}\,u_{t_0t_0} + u_{t_1}\,u_{t_0t_2} - u_{t_2}\,u_{t_0t_1},$$
$$u_{t_1t_4} = u_{t_2t_3} + u_{t_0}\,u_{t_0t_3} - u_{t_3}\,u_{t_0t_1},$$
$$u_{t_2t_4} = u_{t_3t_3} + u_{t_2}\,u_{t_0t_3} - u_{t_3}\,u_{t_0t_2},$$

etc., system $\mathcal{H}_{n-1}$ consists of equations from $\mathcal{H}_{n-2}$ supplemented by equations

$$u_{t_{i-1}t_n} = u_{t_it_{n-1}} + u_{t_0}u_{t_0t_{n-1}} - u_{t_{n-1}}u_{t_0t_{i-1}}, \quad 1 \le i \le n-2.$$

EXAMPLE. Reduced quasi-classical self-dual Yang–Mills equation:

$$u_{yz} = u_{tx} + u_y\, u_{xx} - u_x\, u_{xy} \qquad (\mathcal{E}_3)$$

- E.V. Ferapontov, K.R. Khusnutdinova, 2004

$$\left\{ \begin{array}{lcl} v_t & = & \lambda\, v_y - u_y\, v_x, \\ v_z & = & (\lambda - u_x)\, v_x. \end{array} \right.$$

The symmetry algebra: $\mathrm{Sym}(\mathcal{E}_3) = \mathfrak{q}_\diamond \ltimes \mathfrak{q}_{3,\infty}$,

$\mathfrak{q}_\diamond = \mathfrak{a} \ltimes (\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathfrak{v})$, $\dim \mathfrak{a} = 1$, $\dim \mathfrak{v} = 2$,

$\mathfrak{q}_{3,\infty} = \mathbb{R}_2[h] \otimes \mathbb{R}[t] \otimes \mathfrak{w}[z]$.

The structure equations for $\mathrm{Sym}(\mathcal{E}_3)$:

$$\left\{\begin{array}{lcl} d\alpha & = & 0, \\ d\mathrm{B} & = & \nabla_{h_1}(\mathrm{B}) \wedge \mathrm{B}, \\ d\Gamma & = & \alpha \wedge \Gamma + \nabla_{h_1}(\Gamma) \wedge \mathrm{B} + \frac{1}{2}\, \nabla_{h_1}(\mathrm{B}) \wedge \Gamma, \\ d\Theta & = & \nabla_{h_2}(\Theta) \wedge \Theta + \nabla_{h_1}(\Theta) \wedge (\mathrm{B} + h_0\, \Gamma) \\ & & + h_0\, \nabla_{h_0}(\Theta) \wedge \left(\frac{1}{2}\, \nabla_{h_1}(\mathrm{B}) + h_0\, \nabla_{h_1}(\Gamma) - \alpha\right), \end{array}\right.$$

$$\mathrm{B} = \beta_0 + h_1\, \beta_1 + \frac{1}{2}\, h_1\, \beta_2,$$

$$\Gamma = \gamma_0 + h_1\, \gamma_1,$$

$$\Theta = \sum_{k=0}^{2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i!\, j!}\, h_0^k\, h_1^i\, h_2^j\, \theta_{k,i,j}.$$

$H^1(\mathrm{Sym}(\mathcal{E}_3)) = \mathbb{R}\,\alpha,$

$$H^2_{c\alpha}(\mathfrak{q}_\diamond) = \begin{cases} \mathbb{R}\,[\gamma_0 \wedge \gamma_1], & c = 2, \\ \{[0]\}, & c \neq 2. \end{cases}$$

$H^2_{\alpha}(\mathfrak{q}_\diamond) \subseteq H^2_{\alpha}(\mathrm{Sym}(\mathcal{E}_3)),$

$\Longrightarrow$

we have the non-central extension for $\mathrm{Sym}(\mathcal{E}_2)$ with the additional structure equation

$$d\sigma = 2\,\alpha \wedge \sigma + \gamma_0 \wedge \gamma_1.$$

Integration: $\theta_{2,0,0}$ is a multiple of the contact form $\Rightarrow$

$\sigma - \gamma_1 - \theta_{1,0,0}$ defines the Lax representation

$$\begin{cases} v_t = s\,v_y - u_y\,v_x - \frac{1}{2}\,s^2, \\ v_z = (s - u_x)\,v_x \end{cases}$$

over the rqsdYM.

Replace

$$\mathfrak{q}_\diamond \ltimes (\mathbb{R}_2[h] \otimes \mathbb{R}[t] \otimes \mathfrak{w}[z]) \quad \mapsto \quad \mathfrak{q}_\diamond \ltimes (\mathbb{R}_n[h] \otimes \mathbb{R}[t] \otimes \mathfrak{w}[z]),$$

integrate the same structure equations with

$$\Theta = \sum_{k=0}^{n} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{i!\, j!}\, h_0^k\, h_1^i\, h_2^j\, \theta_{k,i,j},$$

rename $t \mapsto y_1$, $x \mapsto t_0$, $y \mapsto y_0$, $z \mapsto t_1 \implies$

the Lax representation

$$\left\{ \begin{array}{lcl} v_{y_1} & = & s\, v_{y_0} - u_{y_0}\, v_{t_0} - \frac{1}{2}\, s^2 \\ v_{t_1} & = & (s - u_{t_0})\, v_{t_0}, \\ v_{t_2} & = & (s^2 - s\, u_{t_0} - u_{t_1})\, v_{t_0}, \\ & & \dots \\ v_{t_i} & = & \left(s^i - \sum\limits_{j=0}^{i-1} s^{i-j-1}\, u_{t_j} \right) v_{t_0}, \\ & & \dots \\ v_{t_{n-1}} & = & \left(s^{n-1} - s^{n-2}\, u_{t_0} - s^{n-3}\, u_{t_1} - \dots - s\, u_{t_{n-3}} - u_{t_{n-2}} \right)\, v_{t_0}. \end{array} \right.$$

Compatibility conditions: $\mathcal{H}_{n-1}$ supplemented by the rqsdYM equation

$$u_{t_0 y_1} = u_{t_1 y_0} - u_{y_0}\, u_{t_0 t_0} + u_{t_0}\, u_{t_0 y_0}$$

and system

$$u_{t_i y_1} = u_{t_{i+1} y_0} - u_{y_0}\, u_{t_0 t_i} + u_{t_i} u_{t_0 y_0}, \quad 0 \le i \le n-2.$$

$\diamond$

EXAMPLE. The 4D universal hierarchy equation:

$$u_{zz} = u_{tx} + u_z u_{xy} - u_x u_{yz} \qquad (\mathcal{E}_4)$$

- L.V. Bogdanov, M.V. Pavlov, 2017,
- M.V. Pavlov, N. Stoilov, 2017.

Lax representation with a non-removable parameter:

$$\begin{cases} v_t &= \lambda^2 v_x - (\lambda u_x + u_z) v_y, \\ v_z &= \lambda v_x - u_x v_y. \end{cases}$$

The symmetry algebra

$$\mathrm{Sym}(\mathcal{E}_4) = \mathfrak{r}_\diamond \ltimes (\mathbb{R}_1[h] \otimes \mathbb{R}[t] \otimes \mathfrak{w}[y])$$

The structure equations:

$$\left\{ \begin{array}{lcl} d\beta_1 & = & 0, \\ d\beta_2 & = & \beta_1 \wedge \beta_2, \\ d\beta_3 & = & 0, \\ d\beta_4 & = & (\beta_3 - \beta_1) \wedge \beta_4, \\ d\beta_5 & = & \beta_3 \wedge \beta_5 - \beta_2 \wedge \beta_4, \\ d\beta_6 & = & (2\,\beta_3 - \beta_1) \wedge \beta_6 + \frac{1}{2}\,\beta_4 \wedge \beta_5. \\ d\Theta_0 & = & \nabla_2(\Theta_0) \wedge \Theta_0 + \nabla_1(\Theta_0) \wedge (\beta_2 + h_1\,\beta_1) \\ d\Theta_1 & = & \nabla_2(\Theta_1) \wedge \Theta_0 + \nabla_2(\Theta_0) \wedge \Theta_1 + (\beta_1 - \beta_3) \wedge \Theta_1 \\ & & +(\beta_2 + h_1\,\beta_1) \wedge \nabla_1(\Theta_1) + (\beta_5 + h_1\,\beta_4) \wedge \nabla_1(\Theta_0). \end{array} \right.$$

The second non-central extension of $\mathfrak{r}_\diamond$ provides the above covering over 4D UH equation.

Hierarchy: replace $\mathbb{R}_1[h]$ by $\mathbb{R}_n[h]$, extend the structure equations correctly, then integrate $\Rightarrow$

$$
\left\{
\begin{array}{lcl}
v_{y_1} & = & s\,v_{y_0} - u_{y_0}\,v_{t_0}\,, \\
v_{y_2} & = & s^2\,v_{y_0} - (u_{y_1} + s\,u_{y_0})\,v_{t_0}\,, \\
v_{t_1} & = & (s - u_{t_0})\,v_{t_0}\,, \\
v_{t_2} & = & (s^2 - s\,u_{t_0} - u_{t_1})\,v_{t_0}\,, \\
 & & \dots \\
v_{t_i} & = & \left(s^i - \displaystyle\sum_{j=0}^{i-1} s^{i-j-1}\,u_{t_j}\right)\,v_{t_0}\,, \\
 & & \dots \\
v_{t_{n-1}} & = & \left(s^{n-1} - s^{n-2}\,u_{t_0} - s^{n-3}\,u_{t_1} - \dots - s\,u_{t_{n-3}} - u_{t_{n-2}}\right)\,v_{t_0}\,.
\end{array}
\right.
$$

Compatibility conditions: $\mathcal{H}_{n-1}$ supplemented by the 4D UH equation

$$u_{y_0 y_2} = u_{y_1 y_1} + u_{y_0}\, u_{t_0 y_1} - u_{y_1}\, u_{t_0 y_0}$$

and system

$$u_{t_k y_0} = u_{t_{k-2} y_2} + u_{y_1}\, u_{t_0 t_{k-2}} - u_{t_{k-2}}\, u_{t_0 y_1} + u_{y_0}\, u_{t_0 t_{k-1}} - u_{t_{k-1}}\, u_{t_0 y_0},$$

$$u_{t_m y_1} = u_{t_{m-1} y_2} + u_{y_1}\, u_{t_0 t_{m-1}} - u_{t_{m-1}}\, u_{t_0 y_1}$$

with $2 \le k \le n-1$, $1 \le m \le n-1$.

◇

EXAMPLE. The 4D Martínez Alonso–Shabat equation.

$$u_{ty} = u_z\, u_{xy} - u_y\, u_{xz} \qquad (\mathcal{E}_5)$$

Lax representation with non-removable parameter (O.M., 2014):

$$\left\{ \begin{array}{lcl} v_y & = & \lambda\, u_y\, v_x, \\ v_z & = & \lambda\, (u_z\, v_x - v_t). \end{array} \right.$$

Symmetry algebra:

$\mathrm{Sym}(\mathcal{E}_5) = \mathfrak{s}_{2,1} = \mathfrak{s}_{2,1,\infty} \rtimes \mathfrak{s}_{\diamond}$,

$\mathfrak{s}_{2,1,\infty} = (\mathbb{R}_2[h_0] \otimes \mathbb{R}[t] \otimes \mathfrak{w}[x]) \oplus (\mathbb{R}[z] \otimes \mathfrak{w}[y])$

$\mathfrak{s}_{\diamond} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$ (direct sums of two non-abelian 2D Lie algebras).

Denote:

$$\Theta = \sum_{k=0}^{1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p!\,q!}\, h_1^p\, h_2^q\, h_5^k\, \theta_{k,p,q},$$

$$\Omega = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p!\,q!}\, h_3^p\, h_4^q\, \omega_{p,q}.$$

Structure equations:

$$
\begin{aligned}
&d\alpha_0 = 0, \qquad d\alpha_1 = \alpha_0 \wedge \alpha_1,\\
&d\beta_0 = 0, \qquad d\beta_1 = \beta_1 \wedge \beta_2,\\
&d\Theta = \nabla_{h_2}(\Theta) \wedge \Theta + \nabla_{h_1}(\Theta) \wedge (\alpha_1 + h_1\,\alpha_0 - h_5\,\beta_1)\\
&\qquad\qquad + h_5\,\nabla_{h_5}(\Theta) \wedge (\alpha_0 - \beta_0),\\
&d\Omega = \nabla_{h_3}(\Omega) \wedge \Omega + \nabla_{h_4}(\Omega) \wedge (\beta_1 + h_4\,\beta_0).
\end{aligned}
$$

Non-central extension via $H^2_{c_1\alpha_0+c_2\beta_0}(\mathfrak{s}_\diamond)$:

$$
\begin{aligned}
d\gamma_1 &= \alpha_0 \wedge \gamma_1 + \alpha_0 \wedge \alpha_1,\\
d\gamma_2 &= \alpha_0 \wedge \gamma_2 + \beta_0 \wedge \alpha_1,\\
d\gamma_3 &= \beta_0 \wedge \gamma_3 + \alpha_0 \wedge \beta_1,\\
d\gamma_4 &= \beta_0 \wedge \gamma_4 + \beta_0 \wedge \beta_1,\\
d\gamma_5 &= (\alpha_0 + \beta_0) \wedge \gamma_5 + \alpha_1 \wedge \alpha_2.
\end{aligned}
$$

The Wahlquist–Estabrook form

$$\gamma_4 - \gamma_3 + \alpha_1 - \beta_1 - \theta_0 - \omega =$$

$$dv - v_s\, ds - v_t\, dt - v_x\, dx - s\, u_y\, v_x\, dy - (s\,(u_z v_x - v_t) - \ln s)\, dz)$$

for the Lax representation

$$\left\{ \begin{array}{lcl} v_y & = & s\, u_y\, v_x, \\ v_z & = & s(u_z\, v_x - v_t) - \ln s. \end{array} \right.$$

Replace $\mathfrak{s}_{\infty,2,1}$ by $\mathfrak{s}_{\infty,n,m}$, that is, put

$$\Theta = \sum_{k=0}^{n} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p!\,q!}\, h_0^k\, h_1^p\, h_2^q\, \theta_{k,p,q},$$

$$\Omega = \sum_{k=0}^{m} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p!\,q!}\, h_0^k\, h_3^p\, h_4^q\, \omega_{k,p,q},$$

define the structure equations as

$$\left\{ \begin{array}{lcl} d\Theta & = & \nabla_{h_2}(\Theta) \wedge \Theta + \nabla_{h_1}(\Theta) \wedge (\alpha_1 + h_1\,\alpha_0 - h_0\,\beta_1) \\ & & +h_0\,\nabla_{h_0}(\Theta) \wedge (\alpha_0 - \beta_0), \\ d\Omega & = & \nabla_{h_3}(\Omega) \wedge \Omega + \nabla_{h_4}(\Omega) \wedge (\beta_1 + h_4\,\beta_0 - h_0\,\alpha_1) \\ & & +h_0\,\nabla_{h_0}(\Omega) \wedge (\beta_0 - \alpha_0). \end{array} \right.$$

The Wahlquist–Estabrook form

$$\gamma_4 - \gamma_3 + \alpha_0 - \beta_0 - \theta_{n-1} + \sum_{k=0}^{n-3} (n-k-2)\,\theta_k - \sum_{k=0}^{n-1} (n-k)\,\omega_k =$$

$$dv - v_s\,ds - v_t\,dt - v_{x_0}\,dx_0 - \left(s\,(u_z\,v_{x_0} - v_t) - \ln s\right)\,dz$$

$$- \sum_{k=0}^{n-1} \left(\sum_{j=0}^{k} s^{k-j}\,u_{y_j} \right) v_{x_0}\,dy_k$$

$$- \sum_{m=1}^{n-1} \left(s^{-m} - \sum_{j=0}^{m-1} s^{1-m-j}\,u_{x_j} \right) v_{x_0}\,dx_m$$

of the Lax representation ...

$$
\begin{cases}
v_{x_1} & = \left(s^{-1} - u_{x_0}\right)\, v_{x_0}, \\
 & \dots \\
v_{x_m} & = \left(s^{-m} - \sum\limits_{j=0}^{m-1} s^{1-m-j}\, u_{x_j}\right)\, v_{x_0}, \\
 & \dots \\
v_{x_{n-1}} & = \left(s^{1-n} - s^{2-n}\, u_{x_0} - s^{3-n}\, u_{x_1} - \dots - u_{x_{n-2}}\right)\, v_{x_0}, \\
v_{y_0} & = s\, u_{y_0}\, v_{x_0}, \\
 & \dots \\
v_{y_k} & = \left(\sum\limits_{j=0}^{k} s^{k+1-j}\, u_{y_j}\right)\, v_{x_0}, \\
 & \dots \\
v_{y_{n-1}} & = \left(\sum\limits_{j=0}^{k} u_{y_{n-1}} + s\, u_{y_{n-2}} + \dots + s^{n-2}\, u_{y_0}\right)\, v_{x_0}, \\
v_z & = s\,(u_z\, v_{x_0} - v_t) - \ln s
\end{cases}
$$

for the hierarchy that includes system $\mathcal{H}_{n-1}$ as well as equation

$$u_{ty_0} = u_z\, u_{x_0y_0} - u_{y_0}\, u_{x_0z},$$

systems

$$u_{tx_k} = -u_{x_{k+1}z} + u_z\, u_{x_0x_k} - u_{x_k}\, u_{x_0z},$$

$$u_{ty_m} = -u_{y_{m-1}z} + u_z\, u_{x_0y_m} - u_{y_m}\, u_{x_0z},$$

$$u_{x_{k+1}y_0} = u_{y_0}\, u_{x_0x_k} - u_{x_k}\, u_{x_0y_0},$$

$$u_{x_{k+1}y_m} = u_{x_ky_{m-1}} + u_{y_m}\, u_{x_0x_k} - u_{x_0}\, u_{x_ky_m}$$

$$u_{y_0y_k} = u_{y_0}\, u_{x_0y_{k+1}} - u_{y_{k+1}}\, u_{x_0y_0}$$

with $k \in \{0, \ldots, n-2\}$, $m \in \{1, \ldots, n-1\}$, and system

$$u_{y_iy_j} = u_{y_{i-1}y_{j+1}} + u_{y_i}\, u_{x_0y_{j+1}} - u_{y_{j+1}}\, u_{x_0y_i}$$

with $i \in \{1, \ldots, n-2\}$, $j \in \{i, \ldots, n-2\}$.

$\diamond$