# Hexagonal webs, geodesic flows on surfaces, and integrable billiards

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## 1 Introduction

Having published his last book on the web theory [B-55], Blaschke, as was his habit, listed non-solved problems, among them the following, which naturally generalizes one of the most elegant classical results of the web theory, namely, the Graf and Sauer Theorem [GS-24]: find a geometric ctiterion for a metric on a surface to admit a hexagonal geodesic 3-web. The problem was not new and goes back to the end of the 19th century. Finsterwalder [Fi-99] observed that any surface of revolution admits a one-parametric family of hexagonal geodesic 3-webs and asked for more examples, Stäckel e Ahl [St-03] found new examples on the so-called Lie spiral surfaces, Sauer [Sa-26] constructed surfaces of revolution admitting hexagonal geodesic 3-webs of not Finsterwalder type (i.e. not symmetric by rotations), finally, Mayrhofer [Ma-31] showed that the family of hexagonal geodesic 3-webs on a fixed surface is at most 9-parametric, and this is sharp and realized on surfaces of constant gaussian curvature.

It is remarkable, that this old problem on hexagonal geodesic 3-webs is closely related to another classical problem, namely that of integrability of the geodesic flow by first integrals, polynomial in momenta (or, equivalently, in velocity). Namely, in [A-19] the proponent proved that a surface carries a hexagonal 3-web of geodesics if and only if the geodesic flow on the surface admits a cubic first integral with real roots. The relation of the web and the integral is quite natural: the web directions coincide with the zero directions of the integral. This result opens a new geometric perspective for exploring the integrability of geodesic flows by web theory methods.

In more detail, consider a first integral of the geodesic flow, homogeneous and polynomial of degree $n$ in momenta $(p, q)$ :

$$I = \sum_{i=0}^{n} a_i(u,v)p^{n-i}q^i, \tag{1}$$

where $u, v$ are local coordinates on the surface. A direction $[p : q] \in \mathbb{RP}^1$ is a real root of a homogeneous polynomial $I \in \mathbb{R}[p, q]$, if $I(p, q) = 0$. Due to the canonical isomorphism between the tangent and cotangent spaces of Riemannian manifolds, we can view a real root of a polynomial integral of geodesic flow as a direction in the tangent plane, i.e. we get an implicit ODE on the surface:

$$\sum_{i=0}^{n} \tilde{a}_i(u,v)du^{n-i}dv^i = 0.$$

Integrating this implicit ODE we get an $n$-web.

In fact, this approach proved its fruitfulness in studying geometric properties of quadratic integrals on surfaces (see [A-21]). If a quadratic integral has 2 real roots, we obtain a geodesic net $\mathcal{G}_2$ given by integral curves of quadratic implicit ODE, corresponding to the quadratic integral (1) for $n = 2$. Now we use the metric to define the *bisector net* $\mathcal{B}_2$ of the geodesic net $G_2$ as being formed by the integral curves of directions bisecting the angles at which the geodesics of the net $\mathcal{G}_2$ intersect. By construction, the net $\mathcal{B}_2$ is orthogonal. Superposition of $\mathcal{G}_2$ and $\mathcal{B}_2$ gives a 4-web. Then, a geodesic net $\mathcal{G}_2$ corresponds to a quadratic integral if and only if at least one (and therefore any) 3-subweb of the constructed 4-web is hexagonal. It is interesting that the integrability by quadratic integrals has also a conformal translation: it is equivalent to the condition that the above constructed bisector net $\mathcal{B}_2$ is isothermal.

The configuration of a geodesic net $\mathcal{G}_2$ and its bisector net $\mathcal{B}_2$ brings about another interesting point. Consider a leaf $\mathcal{B}$ of the bisector net and two geodesic rays, outbound from a point $m \in \mathcal{B}$ along the leaves of the geodesic net so that $\mathcal{B}$ does not locally separate these rays. One can view this picture either as the light ray reflecting in the mirror $\mathcal{B}$ or as the trajectory of a billiard ball hitting the wall $\mathcal{B}$ at $m$ (see an excellent introduction to the subject [Ta-05]). If the geodesic net is defined by a quadratic integral then this billiard is integrable in any of the two common definitions: either as the existence of an integral of motion, globally or locally near the the billiard wall or as the existence of a smooth foliation by caustics of the billiard table, globally or locally in a neighbourhood of the billiard wall (see the survey [BM-18] for the references and recent advances).

Last decades have seen a resurgence of interests to the classical mathematical game of billiards (see [PT-03, Ta-05, DR-11, IT-17, GIT-19, BM-18] and references therein). The local integrability of billiards bounded by a leaves of Liouville net is not new, moreover, it was shown in [IT-17] that a billiard inside a parallelogram bounded by such leaves possesses the Poncelet property (see monograph [DR-11] on the Poncelet porism). There are also globally integrable billiards with smooth boundaries on surfaces with Liouville metric (see [PT-03]). We construct such billiards on surfaces with three-dimensional projective algebra and prove that the Poncelet porism holds true.

The results obtained in [A-21] generalize the case of flat metrics [A-20], as well as the classical billiards in confocal conics. The "hexagonality property" of integrable billiards naturally supplements four equivalent properties of integrable billiards [GIT-19] by a fifth one.

The problem of geodesic flow integrability by polynomial integrals dates from the 19th century classical works of Dini, Darboux and Koenigs [Di-69, Da-91, Ko-96], when the local theory for the case of linear and quadratic integrals were developed (see also [Ko-82] for a modern treatment and the review [BMF-98] for more references).

The case of cubic integrals is much more involved computationally (see for example [Kr-08, MS-11, VDS-15]) and even a local theory is not well understood yet, for example, we do not know the possible dimensions of the space of cubic integrals, though the local existence of the metric admitting polynomial integrals of *any* degree follows from Cauchy-Kowalevskaya theorem for the PDE system governing the metric and the integral (see [Te-97]).

Finally, we mention that the recent interest to geodesic webs is motivated also by freeform architecture [DPW-11, PHD-10].

## 2 Plan of the minicourse

1. Planar webs, Blaschke curvature, examples of hexagonal webs.
2. Linear webs, web rank, theorem of Graf & Sauer, algebraic webs.
3. Blaschke problem on hexagonal geodesic 3-webs, examples.
4. Geodesic flows on surfaces integrable by integrals polynomial in momenta.
5. Quadratic integrals of geodesic flows and hexagonal webs, integrable billiards, Poncelet theorem for Liouville metrics.

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