THE THEORY OF LINEAR G-DIFFERENCE EQUATIONS

PER K. JAKOBSEN
VALENTIN V. LYCHAGIN

Abstract. We introduce the notion of difference equation defined on a structured set. The symmetry group of the structure determines the set of difference operators. All main notions in the theory of difference equations are introduced as invariants of the action of the symmetry group. Linear equations are modules over the skew group algebra, solutions are morphisms relating a given equation to other equations, symmetries of an equation are module endomorphisms and conserved structures are invariants in the tensor algebra of the given equation.

We show that the equations and their solutions can be described through representations of the isotropy group of the symmetry group of the underlying set. We relate our notion of difference equation and solutions to systems of classical difference equations and their solutions and show that out notions incluse these as a special case.

Date: December 17l, 1997.
1991 *Mathematics Subject Classification.* Primary: 39A05; Secondary: 39A70.
Key words and phrases. Finite difference equations,modules,morphisms,categories.

Contents

1. Introduction

Let us consider a general second order difference equation of the form

$$a_i f_{i+1} + b_i f_i + c_i f_{i-1} = 0$$

Introduce the simple graph $\mathcal{S}$ consisting of vertices $\{x_i\}_{i\in\mathbb{Z}}$ and edges $\{\{x_i, x_{i+1}\}\}_{i\in\mathbb{Z}}$. Let $\mathcal{F}(\mathcal{S})$ be the $\mathbb{R}$-algebra of $\mathbb{R}$-valued functions on the graph $\mathcal{S}$. Then the sequences $\{a_i\}, \{b_i\}, \{c_i\}$ and $\{f_i\}$ are all elements in $\mathcal{F}(\mathcal{S})$. Denote these elements by a,b,c,and f. Let s be the operator of left translation on the lattice $\mathcal{S}$, $sx_i = x_{i-1}$. Then s acts on $\mathcal{F}(\mathcal{S})$ in the natural way

$$(sf)(x_i) = f(s^{-1}x_i)$$

Define $\triangle = as + be + cs^{-1}$ where e acts as the identity on $\mathcal{S}$. Then $\triangle$ acts on $\mathcal{F}(\mathcal{S})$ as a $\mathbb{R}$-linear operator and our original equation can be written

$$\triangle(f) = 0$$

In order to understand what $\triangle$ is in algebraic terms we need to introduce some new notions. Let $G = Aut(\mathcal{S})$ be the automorphism group of the graph $\mathcal{S}$. This group acts on $\mathcal{F}(\mathcal{S})$ in the natural way $(gf)(x_i) = f(g^{-1}x_i)$. Let A be the set of finite formal linear combinations of elements in G with coefficients in $\mathcal{F}(\mathcal{S})$.

$$A = \{\sum_g a_g g \mid a_g \in \mathcal{F}(\mathcal{S})\}$$

On the set A we define addition and scalar multiplication with elements $r \in \mathbb{R}$ componentwise. Product is defined in the following way

$$(ag)(bg') = (agb)(gg')$$

With these operations A is a $\mathbb{R}$-algebra. $A = \mathcal{F}(\mathcal{S})[G]$ is the skew group algebra of G over $\mathcal{F}(\mathcal{S})$. This algebra acts on $\mathcal{F}(\mathcal{S})$ through

$$(\sum_g a_g g)f = \sum_g a_g g(f)$$

Using these notions we observe that our classical difference operator $\triangle = as + be + cs^{-1}$ is an element of the skew group algebra A. It is now evident that we can interpret all elements in A as difference operators over $\mathcal{S}$. We will in fact define A to be the algebra of G-difference operators over $\mathcal{S}$. This means that the notion of difference operator is defined in terms of the symmetries of the underlying graph $\mathcal{S}$. The group of symmetries of $\mathcal{S}$ measures the arbitrariness in the description of $\mathcal{S}$. Without this arbitrariness difference operators could not exist, in a totally asymmetrical space with trivial symmetry group there could be no difference operators and as a consequence no difference equations.

We will in this paper generalize these simple observations and consider a set $\mathcal{S}$ and a group G acting on $\mathcal{S}$. For any such group action there exists some structure on $\mathcal{S}$ such that G is a subgroup of the full automorphism group of this structure. If the set is finite then the group is actually the full group of automorphisms of the space $\mathcal{S}$. The algebra of scalar difference operators on $\mathcal{S}$ will be the skew group algebra $A = \mathcal{F}(\mathcal{S})[G]$ where $\mathcal{F}(\mathcal{S})$ will be the algebra of $\mathbb{F}$-valued functions defined on $\mathcal{S}$. Difference equations on $\mathcal{S}$ and their solutions must be invariant objects under the action of the group G. If they are not invariant their description and solutions will depend on the arbitrariness in the specification of the underlying space. The Klein Erlanger program in geometry has shown that the building blocks of the geometry of a set with a group action are the invariants of the group. Geometrical objects and their relations are constructed from invariants. In this way the geometry will not depend on the arbitrariness of the underlying space. What we propose in this paper is in the spirit of the Erlanger program in geometry.

We propose that the building blocks of the theory of difference equations on a finite space with some structure are the invariants of the group of automorphisms acting on the space. The algebra of difference operators will be the skew group algebra,A, of G and all main notions in the theory of difference equations will be defined in terms of invariants. A linear difference equations $\mathcal{E}$ will be an A-module, symmetries of $\mathcal{E}$ will be A-endomorphisms of $\mathcal{E}$. All conserved quantities and structures of the equations will be invariant elements in the tensoralgebra of the equation $\mathcal{E}$. A special role will be played by the equations corresponding to indecomposable and simple A-modules.

In this paper we introduce a Categorical point of view on equations and solutions. The equations are objects in a full subcategory of the Category of A-modules. Solutions of an equation are descriptions of the equation in terms of other equations.

Only descriptions that are invariants are allowed and this leads to the idea of a solution of an equation in a Category of A-modules as a morphism between the given equation and some other equation. So solutions are morphisms in the Category. Solving an equation thus means to find the G-invariant descriptions or morphisms between the given equation and all other equations in the Category. In this way symmetries are special types of solutions, they are descriptions of an equation in terms of itself. Simple equations play a special role in that they can only be described in terms of themselves. They play the role of atoms in our category of equations. In the semisimple situation all equations are sums of simple equations so the description of a given equation in terms of simple equations in fact given a complete description of the equation. In a more general situation we also need descriptions in terms of indecomposable equations in order to give a complete description of a given equation. The indecomposable equations that are not simple are closely related to the notion of quantization. The family of simple and indecomposable equations is determined by the group of symmetries of the underlying space so this group determines the type of solutions that are needed to solve any equation in the Category. Note that from this point of view a solution is a relational concept. It does not belong to one object but to a pair of objects.

We will in this paper develope the theory for a class of equations we call finite type. These are analogs of the finite type or Frobenius equations in the theory of differential equations. Note that if the set $\mathcal{S}$ is finite then all equations are of finite type. In an upcoming work the theory will be developed for a much wider class of equations.

There exists currently several geometric-algebraic approaches to the study of difference and differential equations; the differential algebra approach of Ritt [1] and Kolchin [2] and the description through use of jet bundles and D-modules [3] just to mention two. Our approach does not belong directly to any of these directions. It is however somewhat related to the approach in [3] and the difference algebra approach in [4].

2. The main notions in the theory of finite type difference equations on a set

Let $\mathcal{S}$ be a set and let G be a group acting on $\mathcal{S}$. We will assume that the action of G is from the left and is faithful and transitive so that G is acting as a transitive group of permutations on $\mathcal{S}$. It is well know from the theory of permutation groups that there exists a finite set of relations on $\mathcal{S}$ such that the group G is included in the full group of symmetries of these relations. A space is a set with some structure defined. Any group acting on a set can thus be thought of a the symmetry group of a space. Examples of such spaces are graphs, lattices, finite projective spaces, finite linear spaces etc.

Example. Let $\mathcal{S}$ be the cyclic graph with vertex set $\{x_1, x_2, \cdots, x_n\}$ and edge set $\{\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_n, x_1\}\}$. This graph can be considered to be a discrete approximation to the circle S^1. The group of symmetries of this finite space is the dihedral group, D_{2n}. It has two generators t and s, where t is reflection around x_1 and s is left translation. The symmetry group has $2n$ elements and presentation in terms of generators and relations in the following form, $D_{2n} = \langle s, t \mid s^n = 1, t^2 = 1, tst = s^{n-1} \rangle$.

We will now define the main notions in the theory of difference equations on a space S as the invariants associated to the group of symmetries of S.

2.1. **The algebra of G-difference operators.** Let $\mathbb{F}$ be a field and let $\mathcal{F}(\mathcal{S})$ be the $\mathbb{F}$-algebra of $\mathbb{F}$ valued functions on S. Let G be the symmetry group of the space S. The the left action of G on S can be lifted to a left action of G on $\mathcal{F}(\mathcal{S})$ in the natural way

$$(g(f))(x) = f(s^{-1}x) \quad \forall g \in G$$

The skew group algebra of G over $\mathcal{F}(\mathcal{S})$ is the set of finite formal linear combinations of elements of G with coefficients in $\mathcal{F}(\mathcal{S})$. Addition and multiplication by elements in $\mathbb{F}$ is defined componentwise and multiplication is defined by

$$(fg)(hg') = (fg(h))gg'$$

We now define the first basic notion in our theory of difference equations.

Definition 1. $\mathcal{F}(\mathcal{S})[G]$ is the algebra of *G-difference operators* on the set S.

Notation. We will from now on use the notation $k = \mathcal{F}(\mathcal{S})$ and $A = \mathcal{F}(\mathcal{S})[G]$.

Example. Let $\mathcal{S}$ be the cyclic graph $\mathcal{S}$ with n elements. We have seen that the symmetry group of $\mathcal{S}$ is the dihedral group D_{2n} with generators beeing left translation s and reflection t. The algebra of D_{2n}-difference operators consists of formal linear combinations of $\mathbb{F}$-valued functions and elements of D_{2n}. The algebra A contains the usual difference operators from the calculus of differences whose continum limit corresponds to the usual ordinary differential operators. But it also contains operators involving the reflection t. These operators will in the continum limit correspond to differential-delay equations.

2.2. **Linear G-difference equations of finite type and solutions.** Let A be the algebra of G-difference operators on a space $\mathcal{S}$. Let $\mathcal{E}$ be a finitely generated module over k. If not otherwise noted finitely generated means finitely generated over k. Assume that G acts on $\mathcal{E}$ on the left, $g(fe) = g(f)ge$. Then $\mathcal{E}$ is a left A-module with the natural action of the skew group algebra A on $\mathcal{E}$. In this way $\mathcal{E}$ can be considered to be an invariant for the symmetry group G of the underlying space. We will consider only left A-modules that can be given a geometrical interpretation. Define $\mu_x \subset k$ by

$$\mu_x = \{f : \mathcal{S} \to \mathbb{F} \mid \ f(x) = 0\}$$

The subsets μ_x are clearly ideals in k. They are in fact maximal ideals

Proposition 1. *μ_x is a maximal ideal in $\mathcal{F}(\mathcal{S})$.*

Proof. μ_x is clearly an ideal in $\mathcal{F}(\mathcal{S})$. Let J be an ideal in $\mathcal{F}(\mathcal{S})$ and assume that $\mu_x \subset J \subset \mathcal{F}(\mathcal{S}) \Rightarrow \exists\ j \in J$ such that $j \notin \mu_x \Rightarrow j(x) \neq 0 \Rightarrow \frac{j}{j(x)} \in J$. But then $(\frac{j}{j(x)} - 1)(x) = \frac{j(x)}{j(x)} - 1 = 0 \Rightarrow \frac{j}{j(x)} - 1 \in \mu_x$. But $\mu_x \subset J \Rightarrow \frac{j}{j(x)} - 1 \in J \Rightarrow \frac{j}{j(x)} - (\frac{j}{j(x)} - 1) = 1 \in J \Rightarrow J = \mathcal{F}(\mathcal{S})$. This is a contradiction so $J = \mu_x$ and μ_x is maximal. □

For each $x \in \mathcal{S}$ we have a submodule $\mu_x\mathcal{E}$ since μ_x is an ideal. We will only consider left A-modules that have no invisible elements [3].

Definition 2. $\mathcal{E}$ is a *geometric* left A-module iff

$$\bigcap_{x\in\mathcal{S}} \mu_x\mathcal{E} = 0 \tag{2.1}$$

$\mathcal{E}$ beeing a left A-module means that we have an action of the algebra of G-difference operators on $\mathcal{E}$. We are now ready to define the second main notion in our theory.

Definition 3. A linear G-difference equation of finite type is a geometric left A-module that is finitely generated over k.

We will use the term GF-difference equations for the equations defined in the previous definition. In general the structure of a GF-difference equation is investigated by comparing it to other equations. An equation will be considered to be understood only if its relations to all other equations are known. This is the Categorical point of view. Relations between equations are A-morphisms so an equation $\mathcal{E}$ is understood or *solved* if $Hom_A(\mathcal{E},\mathcal{F})$ is known for all GF-difference equations $\mathcal{F}$. Let us formalize this in a definition

Definition 4. Let $\mathcal{E}$ be any GF-difference equation. Then a solution of $\mathcal{E}$ of type $\mathcal{F}$, where $\mathcal{F}$ is a GF-difference equation, is a A-module morphism $\phi \in Hom_A(\mathcal{E},\mathcal{F})$

Using this definition we can now say that a GF-difference equation is solved if we know all solutions of the equation. We will introduce two special types of solutions that will play a central role in our theory. A GF-difference equation is indecomposable if it can not be written as a direct sum of two GF-difference equations. Our first special type of solution is the following

Definition 5. Let $\mathcal{E}$ be any GF-difference equation and let $\mathfrak{S}$ be a indecomposable equation. Then a indecomposable solution of $\mathcal{E}$ of type $\mathfrak{S}$ is an element of $Hom_A(\mathcal{E},\mathfrak{S})$.

The second special type of solution is symmetries. These are relations that describe the equation in terms of itself so we define.

Definition 6. Let $\mathcal{E}$ be any GF-difference equation. Then a symmetry of $\mathcal{E}$ is a A-morphism of $\mathcal{E}$ to itself.

So a symmetry of $\mathcal{E}$ is an element of $End_A(\mathcal{E})$. If $f \in Hom_A(\mathcal{E},\mathcal{F})$ is any solution $\mathcal{E}$ of type $\mathcal{F}$ and ϕ is a symmetry of $\mathcal{E}$ then $\phi^*(f) = \phi\circ f$ is also a solution of type $\mathcal{F}$. So symmetries map solutions of some type to solutions of the same type. The problem of solving an equation is closely linked to the module structure of the equation and we will now start to develope the structure theory for GF-difference equations.

3. The structure of the Category of GF-difference equations

Let $\mathbb{GFE}$ be a category [5] whose objects are GF-difference equations and morphisms are A-module morphisms.

Definition 7. $\mathbb{GFE}$ is the category of GF-difference equations

A complete description of the structure of the category $\mathbb{GFE}$ is the same as knowing all solutions to all GF-difference equations. This is in general an enormously complicated problem. We will in this section describe what can be said in general about the structure of the category $\mathbb{GFE}$ without placing any restrictions on the set $\mathcal{S}$ or the group G. We will start our investigation of the structure of $\mathbb{GFE}$ by investigating the closure of the set of GF-difference with respect to the usual linear algebra operations like direct sum, tensor product etc. These operations preserve the set of modules that are finitely generated over k. They also preserve the property of beeing geometric as we will now see.

3.1. **The algebra of GF-difference equations.** Let $\mathcal{E}_1, \mathcal{E}_2$ be two GF-difference equations. Then $\mathcal{E}_1 \oplus \mathcal{E}_2$ is a finitely generated left A-module with the operations

$$f(e_1, e_2) = (fe_1, fe_2) \quad \forall f \in k$$
$$g(e_1, e_2) = (ge_1, ge_2) \quad \forall g \in G.$$

Proposition 2. *The direct sum of GF-difference equations $\mathcal{E}_1 \oplus \mathcal{E}_2$ is a GF-difference equation.*

Proof. We know that $\bigcap_{x \in \mathcal{S}} \mu_x \mathcal{E}_1 = 0$, $\bigcap_{x \in \mathcal{S}} \mu_x \mathcal{E}_2 = 0$ and we have by definition that $\mu_x \mathcal{E} = \{\sum_i f_i e_i \mid f_i \in \mu_x, e_i \in \mathcal{E}\} \Rightarrow \mu_x(\mathcal{E}_1 \oplus \mathcal{E}_2) = \{\sum_i f_i(e_i^1 + e_i^2) \mid f_i \in \mu_x, e_i^1 \in \mathcal{E}_1, e_i^2 \in \mathcal{E}_2\}$. But $\sum_i f_i(e_i^1 + e_i^2) = \sum_i f_i e_i^1 + \sum_i f_i e_i^2 \in \mu_x \mathcal{E}_1 \oplus \mu_x \mathcal{E}_2$. So $\mu_x(\mathcal{E}_1 \oplus \mathcal{E}_2) \subset \mu_x \mathcal{E}_1 \oplus \mu_x \mathcal{E}_2$. $\Rightarrow \bigcap_{x \in \mathcal{S}} \mu_x(\mathcal{E}_1 \oplus \mathcal{E}_2) \subset \bigcap_{x \in \mathcal{S}} (\mu_x \mathcal{E}_1 \oplus \mu_x \mathcal{E}_2) \subset \bigcap_{x \in \mathcal{S}} \mu_x \mathcal{E}_1 \oplus \bigcap_{x \in \mathcal{S}} \mu_x \mathcal{E}_2 = 0 + 0 = 0$. So the direct sum is a finitely generated geometric left A-module. □

Let $\mathcal{E}_1, \mathcal{E}_2$ be GF-difference equations. From this it follows that they are left k-modules since $k \subset A$ as algebras. The algebra k is abelian so $\mathcal{E}_1 \otimes_k \mathcal{E}_2$ is a well defined finitely generated k-module. Define a G action on the tensorproduct module by $g(e_1 \otimes_k e_2) = ge_1 \otimes_k ge_2$. With this action we have

Proposition 3. *$\mathcal{E}_1 \otimes_k \mathcal{E}_2$ is a A-module with the given G action.*

Proof.

$$\begin{aligned}
g(fe_1 \otimes_k e_2) &= g(fe_1) \otimes_k ge_2 = g(f)(ge_1) \otimes_k ge_2 \\
&= ge_1 \otimes_k g(f)(ge_2) = ge_1 \otimes_k g(fe_2) \\
&= g(e_1 \otimes_k fe_2). \\
(g_1 g_2)(e_1 \otimes_k e_2) &= (g_1 g_2)e_1 \otimes_k (g_1 g_2)e_2 = g_1(g_2 e_1) \otimes_k g_1(g_2 e_2) \\
&= g_1(g_2 e_1 \otimes_k g_2 e_2) = g_1(g_2(e_1 \otimes_k e_2)). \\
(gf)(e_1 \otimes_k e_2) &= g(fe_1 \otimes_k e_2) = g(fe_1) \otimes_k ge_2 \\
&= (g(f)g)e_1 \otimes_k ge_2 = (g(f))(ge_1 \otimes_k ge_2) = (g(f)g)(e_1 \otimes_k e_2).
\end{aligned}$$

□

Proposition 4. *Assume that $\mathcal{E}_1, \mathcal{E}_2$ are two GF-difference equations. Then $\mathcal{E}_1 \otimes_k \mathcal{E}_2$ is also a GF-difference equation.*

Proof. We observe that $\mu_x(\mathcal{E}_1 \otimes_k \mathcal{E}_2) \subset \mu_x \mathcal{E}_1 \otimes_k \mathcal{E}_2$. So we have $\bigcap_{x \in \mathcal{S}} \mu_x(\mathcal{E}_1 \otimes_k \mathcal{E}_2) \subset \bigcap_{x \in \mathcal{S}} (\mu_x \mathcal{E}_1 \otimes_k \mathcal{E}_2) \subset (\bigcap_{x \in \mathcal{S}} \mu_x \mathcal{E}_1) \otimes_k \mathcal{E}_2 = 0$. □

Let $\mathcal{E}_1, \mathcal{E}_2$ be two GF-difference equations. Then $Hom_k(\mathcal{E}_1, \mathcal{E}_2)$ is a finitely generated left k-module with the natural action of k

$$(f\phi)(e) = f(\phi(e)) \quad \forall f \in k$$

Define an action of G on $Hom_k(\mathcal{E}_1, \mathcal{E}_2)$ by

$$(g\phi)(e) = g(\phi(g^{-1}e)) \quad \forall g \in G$$

Proposition 5. *$Hom_k(\mathcal{E}_1, \mathcal{E}_2)$ is a left A-module with the given action of k and G.*

Proof.

$$\begin{aligned}
(g\phi)(fe) &= g(\phi(g^{-1}fe)) = g(\phi((g^{-1})fg^{-1}e)) = g((g^{-1})f\phi(g^{-1}e)) \\
&= g(g^{-1})f(g\phi(g^{-1}e)) = f((g\phi)(e)) \\
((g_1g_2)\phi)(e) &= (g_1g_2)\phi((g_1g_2)^{-1}e) = g_1(g_2\phi(g_2^{-1}(g_1^{-1}e))) \\
&= g_1((g_2\phi)(g_1^{-1}e)) = (g_1(g_2\phi))(e) \\
((gf)(\phi))(e) &= (g(f\phi))(e) = g((f\phi)(g^{-1}e)) = g(f(\phi(g^{-1}e))) \\
&= g(f)(g(\phi(g^{-1}e))) = g(f)((g\phi)(e)) = (g(f)(g\phi))(e) = ((g(f)g)(\phi))(e).
\end{aligned}$$

□

Proposition 6. *$Hom_k(\mathcal{E}_1, \mathcal{E}_2)$ is GF-difference equation.*

Proof. Let $\psi \in \mu_x Hom_k(\mathcal{E}_1, \mathcal{E}_2) \Rightarrow \psi = \sum_i f_i\phi_i$ with $f_i \in \mu_x, \phi_i \in Hom_k(\mathcal{E}_1, \mathcal{E}_2)$ $\Rightarrow \psi(e) = \sum_i f_i(\phi(e)) \in \mu_x\mathcal{E}_2$ so $\psi(e) \in \mu_x\mathcal{E}_2$ for all $e \in \mathcal{E}_1 \Rightarrow \psi \in Hom_k(\mathcal{E}_1, \mu_x\mathcal{E}_2)$. So we have $\mu_x Hom_k(\mathcal{E}_1, \mathcal{E}_2) \subset Hom_k(\mathcal{E}_1, \mu_x\mathcal{E}_2)$. But then $\bigcap_{x\in\mathcal{S}} \mu_x Hom(\mathcal{E}_1, \mathcal{E}_2) \subset \bigcap_{x\in\mathcal{S}} Hom_k(\mathcal{E}_1, \mu_x\mathcal{E}_2) \subset Hom_k(\mathcal{E}_1, \bigcap_{x\in\mathcal{S}} \mu_x\mathcal{E}_2) = 0$ □

As a special case of the last proposition we have

Corollary 1. *Let $\mathcal{E}$ be a GF-difference equation. Then the dual $\mathcal{E}^*$ is also a GF-difference equation.*

Let us next consider the case of quotients. Assume that $\mathcal{E}$ is a GF-difference equation and let $\mathcal{E}' \subset \mathcal{E}$ be a submodule of $\mathcal{E}$.

Proposition 7. *$\mathcal{E}/\mathcal{E}'$ is GF-difference equation.*

Proof. Since $\mathcal{E}/\mathcal{E}'$ is finitely generated we only need to prove that it is geometric. Let $h \in \mu_x(\mathcal{E}/\mathcal{E}')$ then $h = \sum_i f_i[e_i]$ where $f_i \in \mu_x$ and $[e_i] \in \mathcal{E}/\mathcal{E}'$ are the equivalent classes of elements in $\mathcal{E}$. So $h = \sum_i f_i[e_i] = [\sum_i f_ie_i]$ and we can conclude $\mu_x(\mathcal{E}/\mathcal{E}') \subset \mu_x\mathcal{E}/\mathcal{E}'$. But then we have $\bigcap_{x\in\mathcal{S}} \mu_x(\mathcal{E}/\mathcal{E}') \subset \bigcap_{x\in\mathcal{S}} \mu_x\mathcal{E}/\mathcal{E}' \subset (\bigcap_{x\in\mathcal{S}} \mu_x\mathcal{E})/\mathcal{E}' = 0$. □

We already know that tensor products and direct sums of GF-difference equations are GF-difference equations. This implies that the tensor algebra $T\mathcal{E}$ of a GF-difference equation is a GF-difference equation. The modules $S^n\mathcal{E}$ and $\wedge^n\mathcal{E}$ are factors of the tensor algebra of $\mathcal{E}$ so we have the following result.

Corollary 2. *Let $\mathcal{E}$ be a GF-difference equation. Then $S^n\mathcal{E}$ and $\wedge^n\mathcal{E}$ are GF-difference equations.*

3.2. **GF-difference equations as modules of sections in vectorbundles.** We have seen that the category of GF-difference equations is closed with respect to quotients, $\oplus$,$\otimes_k$,Hom_k, $\wedge_k$ and S^n_k. These modules can be given an interpretation as modules of sections in vectorbundles over the set $\mathcal{S}$.

Let $\mathcal{E}$ be a GF-difference equation. Then in particular $\mathcal{E}$ is a k-module and $\mu_x\mathcal{E} \subset \mathcal{E}$ is a k submodule of $\mathcal{E}$. Let $E_x = \mathcal{E}/\mu_x\mathcal{E}$. Then E_x is A-module over $k/\mu_x \approx \mathbb{F}$ and therefore is a $\mathbb{F}$ vectorspace. Denote the elements of E_x by $[e]_x$ where $[e]_x = [e']_x$ only if $e - e' \in \mu_x\mathcal{E}$. Let the bundle B over $\mathcal{S}$ be defined by

$$B = \bigcup_{x\in\mathcal{S}} (x, E_x)$$

where the projection,$\pi : B \longrightarrow \mathcal{S}$, in the bundle is projection on the first component. Let $\Gamma(B)$ be the set of sections in the bundle B. This set is a module over k through pointwise addition and multiplication by functions in k.

For each element in G define a bundle map in the bundle B through

$$g(x, [e]_x) = (gx, [ge]_{gx}). \tag{3.1}$$

This set of bundle maps in fact defines an action of G on the bundle B.

Proposition 8. *Bbundle map 3.1 is well defined and determines an action of G on the bundle B.*

Proof. Assume $[e]_x = [e']_x$. Then $e - e' \in \mu_x\mathcal{E}$ and so $ge - ge' \in g(\mu_x\mathcal{E})$. Let $\tilde{e} \in \mu_x\mathcal{E}$, then $\tilde{e} = \sum_i f_i e_i$ where $e_i \in \mathcal{E}$ and $f_i \in \mu_x$ and so $g\tilde{e} = \sum_i g(f)_i ge_i$. But $g(f)_i(gx) = f_i(x) = 0$ so $g(f_i) \in \mu_{gx}$. Then it follows that $ge - ge' \in \mu_{gx}\mathcal{E}$ and we can conclude that $[ge]_{gx} = [ge']_{gx}$ so the map is well defined. Using the definition of the bundle map we have

$$\begin{aligned}(g_1g_2)(x, [e]_x) &= ((g_1g_2)x, [(g_1g_2)e]_{(g_1g_2)x}) = (g_1(g_2x), [g_1(g_2e)]_{g_1(g_2x)}) \\ &= g_1(g_2x, [g_2e]_{g_2x}) = g_1(g_2(x, [e]_x))\end{aligned}$$

so the bundle map defines an action of G on the bundle B. □

Corollary 3. *B is a vectorbundle,that is $dim E_x$ is constant.*

Proof. Let $x \in \mathcal{S}$ be a fixed point in the set $\mathcal{S}$. The group acts transitively on the set $\mathcal{S}$ so for any $y \in \mathcal{S}$ there exists a $g \in G$ such that $gx = y$.This element induces a map $\phi_g : E_x \longrightarrow E_y$ defined by $\phi_g([e]_x) = [ge]_{gx}$. This map is linear and has an inverse $\phi_{g^{-1}}$. We can therefore conclude that all fibers E_y of the bundle B have the same dimension so B is a vectorbundle. □

We now induce an action of G on $\Gamma(B)$ defining

$$(gs)(x) = g(s(g^{-1}x)) \tag{3.2}$$

Proposition 9. *Action 3.2 gives $\Gamma(B)$ the structure of an A-module.*

Proof.

$$
\begin{aligned}
((gf)(s))(x) &= (g(fs))(x) = g((fs)(g^{-1}x)) = g(g(f)(x)s(g^{-1}x)) \\
&= g(f)(x)(gs)(x) = (g(f)g)(s)(x) \\
((g_1g_2)s)(x) &= (g_1g_2)s((g_1g_2)^{-1}x) = g_1(g_2s(g_2^{-1}(g_1^{-1}x))) \\
&= g_1((g_2s)(g_1^{-1})) = (g_1(g_2s))(x)
\end{aligned}
$$

□

Proposition 10. *$\mathcal{E} \approx \Gamma(B)$ as A-modules.*

Proof. Let $e \in \mathcal{E}$. Define $\phi(e) \in \Gamma(B)$ by $\phi(e)(x) = [e]_x$. We clearly have $\phi : \mathcal{E} \longrightarrow \Gamma(B)$.

$$
\begin{aligned}
\phi(e+f')(x) &= [e+f']_x = [e]_x + [f']_x = \phi(e)(x) + \phi(f')(x) = (\phi(e)+\phi(f'))(x), \\
\phi(fe)(x) &= [fe]_x = f(x)[e]_x = f(x)\phi(e)(x) = (f\phi(e))(x).
\end{aligned}
$$

We have used that fact that $z[e]_x = [fe]_x$ where f is any function such that $f(x) = z$. This is well defined because if $[e]_x = [e']_x$ and $f(x) = z, f'(x) = z$ then $e' = e+h, f' = f+g$ where $h \in \mu_x\mathcal{E}$ and $g \in \mu_x$. But then $fe - f'e' = fe-(f+g)(e+h) = -ge - fh - gh \in \mu_x\mathcal{E}$. So $[fe]_x = [f'e']_x$.

We have now proved that ϕ is a k-module morphism. It is also a A-module morphism

$$
\phi(ge)(x) = [ge]_x = [ge]_{g(g^{-1}x)} = g([e]_{g^{-1}x}) = g(\phi(e)(g^{-1}x)) = (g\phi(e))(x).
$$

Assume $\phi(e) = \phi(e')$. Then $[e]_x = [e']_x$ so $[e-e']_x = 0 \quad \forall x \in \mathcal{S}$. But this means that $e - e' \in \bigcap_{x\in\mathcal{S}} \mu_x\mathcal{E}$. We can therefore conclude that $e = e'$ because $\mathcal{E}$ is geometric. So ϕ is injective.

For each $y \in \mathcal{S}$ let $\Pi_y : \mathcal{E} \longrightarrow E_y$ be the canonical projection. B is a vectorbundle so that the dimension n of each fibre as a $\mathbb{F}$-vector space is the same. Let $\{e_i\}_{i=1}^m$ be a set of generators for $\mathcal{E}$. Then $\Pi_y(\{e_i\}_{i=1}^m)$ generates E_y for all $y \in \mathcal{S}$ so at each point at least one subset of say n elements of $\{e_i\}_{i=1}^n$ form a basis for E_y after projection by Π_y. There are only finitely many subsets of n elements from the set of m generators. Enumerate these subsets

$$
B^i = \{e_{l(i,k)}\}_{k=1}^n \quad i = 1 \cdots r.
$$

Here $l(i,k)$ is an index function. Put $\mathcal{S}_1 = \mathcal{S}$ and define subsets $V_i \subset \mathcal{S}$ recursively

$$
\begin{aligned}
V_i &= \{y \in \mathcal{S}_i \mid \Pi_y(B^i) \quad \text{is a basis of } E_y\}, \\
\mathcal{S}_{i+1} &= \mathcal{S}_i - V_i.
\end{aligned}
$$

This gives us a finite set of nonempty subsets $\{V_i\}_{i=1}^p$ such that $V_i \bigcap V_j = \emptyset$ for $i \neq j$, $\mathcal{S} = \bigcup_{i=1}^p V_i$ and $\Pi_y B^i = \{[e_{l(i,k)}]_y\}_{k=1}^n$ is a basis for E_y for all $y \in \mathcal{S}$. Let δ_{V_i} be the characteristic function for V_i. Then $\delta_{V_i} \in k$ and $\sum_i \delta_{V_i} = 1$ Let $T_i = \delta_{V_i}\Gamma(B)$. Then T_i is a k-submodule of $\Gamma(B)$ and T_i has k-basis $\{\delta_{V_i}\phi(e_{l(i,k)})\}_{k=1}^n$. Let $s \in \Gamma(B)$

be any section. Then we have

$$s = (\sum_i \delta_{V_i})(s) = \sum_i \delta_{V_i} s = \sum_i \sum_k f_{ik}\delta_{V_i}\phi(e_{l(i,k)})$$
$$= \phi(\sum_i \sum_k f_{ik}\delta_{V_i} e_{l(i,k)}).$$

But $e = \sum_i \sum_k f_{ik}\delta_{V_i} e_{l(i,k)} \in \mathcal{E}$ so ϕ is surjective.

□

This result show that the category $\mathbb{GFE}$ is equivalent to the category of modules of sections in vectorbundles $\Gamma(B)$ over $\mathcal{S}$ where the action of G is defined through 3.1 and 3.2.

Proposition 11. *Let $\mathcal{E}$ be a GF-difference equation. Then $\mathcal{E}$ is free and finitedimensional as a k-module.*

Proof. Let $\{e_i^x\}_{i=1}^n$ be a basis for E_x over $\mathbb{F}$. The number n exists and is independent of x since we all our bundles are finite dimensional and vector bundles so that the dimension of all fibers are the same. Define sections $\{s_i\}_{i=1}^n$ by $s_i(x) = e_i^x$. Assume that $\sum_{i=1}^n f_i s_i = 0$. Then $\sum_{i=1}^n f_i(x)e_i^x = 0$ so $f_i(x) = 0$ for all $x \in \mathcal{S}$ and all i. This implies that $f_i = 0$ for all i and we conclude that $\{s_i\}_{i=1}^n$ is a linearly independent set over k. Let $s \in \Gamma(B)$, then $s(x) = e_x \in E_x$ so there exists complex numbers $\{c_i^x\}_{i=1}^n$ such that $s(x) = \sum_{i=1}^n c_i^x e_i^x$. Define functions in k by $f_i(x) = c_i^x$, then $s = \sum_{i=1}^x f_i s_i$ and $\{s_i\}_{i=1}^n$ is a spanning set. □

We can use this result to prove a standard isomorphism. Define a map $\phi : \mathcal{E}^* \times \mathcal{F} \longrightarrow Hom_k(\mathcal{E}, \mathcal{F})$ by

$$\phi(e^*, f)(e) = e^*(e)f$$

Proposition 12. *ϕ is k-bilinear.*

Proof.

$$\begin{aligned}
\phi(e_1^* + e_2^*, f)(e) &= \phi((e_1+e_2)^*, f)(e) = (e_1+e_2)^*(e)f = e_1^*(e)f + e_2^*(e)f \\
&= \phi(e_1^*, f)(e) + \phi(e_2^*, f)(e) = (\phi(e_1^*, f) + \phi(e_2^*, f))(e) \\
\phi(e^*, f_1 + f_2)(e) &= e^*(e)(f_1 + f_2) = e^*(e)f_1 + e^*(e)f_2 \\
&= (\phi(e_1^*, f_1) + \phi(e^*, f_2))(e) \\
\phi(re^*, f)(e) &= (re^*)(e)f = (r(e^*(e)))f = r(e^*(e)f) \\
&= r(\phi(e^*, f)(e)) = (r\phi(e^*, f))(e)
\end{aligned}$$

□

This proposition show that we have a well defined map $\phi : \mathcal{E}^* \otimes_k \mathcal{F} \longrightarrow Hom_k(\mathcal{E}, \mathcal{F})$ defined by $\phi(e^* \otimes_k f)(e) = e^*(e)f$.

Proposition 13. *ϕ is an A-isomorphism*

Proof. Let $\{e_i\}_{i=1}^n, \{f_i\}_{i=1}^m$ be basis over k for $\mathcal{E}$ and $\mathcal{F}$. Let $\{e_i^*\}_{i=1}^n$ be the dual basis for $\mathcal{E}^*$. Then $\{e_i^* \otimes_k f_j\}$ is a basis for $\mathcal{E}^* \otimes_k \mathcal{F}$ because the modules are free over k. Let $v = \sum_{ij} a_{ij} e_i^* \otimes_k f_j$ and assume that $\phi(v) = 0$. Then $\phi(v)(e_s) = 0$ for all s and we have $\sum_j a_{sj} f_j = 0$ so that $a_{ij} = 0$ for all i and j because $\{f_j\}$ is a basis for $\mathcal{F}$. So ϕ is injective. Let $F \in Hom_k(\mathcal{E}, \mathcal{F})$. Define the matrix (F_{ij})

by $F(e_i) = \sum_j F_{ij} f_j$ and let $v = \sum_{ij} F_{ij} e_i^* \otimes_k f_j$. Then $\phi(v) = F$ so that ϕ is surjective. Finally we have

$$\begin{aligned}\phi(g(e^* \otimes_k f))(e) &= \phi(ge^* \otimes_k gf)(e) = (ge^*)(e)gf = (g(e^*(g^{-1}e)))gf \\ &= g(e^*(g^{-1}e)f) = g(\phi(e^* \otimes_k f)(g^{-1}e)) = (g\phi(e^* \otimes_k f))(e)\end{aligned}$$

so that ϕ is an A-morphism. □

Corollary 4. $Hom_k(\mathcal{E}, \mathcal{F}) \approx \mathcal{E}^* \otimes_k \mathcal{F}$

We know that the category of GF-difference equations is closed with respect to the usual linear algebra operations. Since we have proved that any GF-difference equation is isomorphic to A-module of sections in a vectorbundle over $\mathcal{S}$ it is evident that all such linear algebra operations must reduce to operations on the corresponding vectorbundles. The following series of propositions show that the correspondence is as nice as one would expect.

Let $\mathcal{E}_1, \mathcal{E}_2$ be two GF-difference equations.. Then $\mathcal{E}_1 \approx \Gamma(B_1)$ and $\mathcal{E}_2 \approx \Gamma(B_2)$ where B_1, B_2 are vectorbundles

$$B_1 = \bigcup_{x \in \mathcal{S}} (x, E_x^1)$$
$$B_2 = \bigcup_{x \in \mathcal{S}} (x, E_x^2)$$

We then have

Proposition 14. $\mathcal{E}_1 \oplus \mathcal{E}_2 \approx \Gamma(B_1 \oplus B_2)$.

Proof. Define a map $\phi : \mathcal{E}_2 \oplus \mathcal{E}_2 \longrightarrow \Gamma(B_1 \oplus B_2)$ by

$$\phi(s_1, s_2)(x) = (s_1(x), s_2(x))$$

where we identify the GF-difference equations $\mathcal{E}_1, \mathcal{E}_2$ with their corresponding modules of sections.

Assume $\phi(s_1, s_2) = \phi(s_1', s_2')$. Then $(s_1(x), s_2(x)) = (s_1'(x), s_2'(x))$ and so $s_1(x) = s_1'(x), s_2(x) = s_2'(x)$ for all $x \in \mathcal{S}$. But this implies that $(s_1, s_2) = (s_1', s_2')$ and ϕ is injective.

Let $s \in \mathcal{E}_1 \oplus \mathcal{E}_2$. Then $s(x) \in E_x^1 \oplus E_x^2$ for all $x \in \mathcal{S}$. Define $s_1(x) = \pi_1 \circ s(x)$ and $s_2(x) = \pi_2 \circ s(x)$ where $\pi_1 : E_x^1 \oplus E_x^2 \longrightarrow E_x^1$ and $\pi_2 : E_x^1 \oplus E_x^2 \longrightarrow E_x^2$ are the projections on the first and second factor. But then $(s_1, s_2) \in \mathcal{E}_1 \oplus \mathcal{E}_2$ and evidently

$$\phi(s_1, s_2)(x) = (s_1(x), s_2(x)) = s(x)$$

so ϕ is surjective. Furthermore we have

$$\begin{aligned}\phi(f(s_1, s_2))(x) &= \phi(fs_1, fs_2)(x) = ((fs_1)(x), (fs_2)(x)) \\ &= (f(x)s_1(x), f(x)s_2(x)) = f(x)(s_1(x), s_2(x)) = (f\phi(s_1, s_2))(x), \\ \phi(g(s_1, s_2))(x) &= \phi(gs_1, gs_2)(x) = ((gs_1)(x), (gs_2)(x)) \\ &= (g(s_1(g^{-1}x)), g(s_2(g^{-1}x))) = g((s_1, s_2)(g^{-1}x)) = (g\phi(s_1, s_2))(x).\end{aligned}$$

So ϕ is a left A-module morphism and the proof is complete. □

We have seen that the k-tensor product of GF-difference equations is a GF-difference equation with the action of k and G defined by

$$f(s_1 \otimes_k s_2) = fs_1 \otimes_k s_2 \quad \forall f \in k,$$
$$g(s_1 \otimes_k s_2) = gs_1 \otimes_k gs_2 \quad \forall g \in G.$$

Using the vectorbundles B_1 and B_2 corresponding to $\mathcal{E}_1$ and $\mathcal{E}_2$ we define a new vectorbundle $B_1 \otimes_{\mathbb{F}} B_2$ by

$$B_1 \otimes_{\mathbb{F}} B_2 = \bigcup_{x \in \mathcal{S}} (x, E_x^1 \otimes_{\mathbb{F}} E_x^2).$$

Let $\Gamma(B_1 \otimes_{\mathbb{F}} B_2)$ be the set of sections in the vectorbundle $B_1 \otimes_{\mathbb{F}} B_2$. This set is a k-module through pointwise addition and multiplication by elements of k. It is also a left A-module through the action

$$(gs)(x) = g(s(g^{-1}x))$$

where

$$g(x, [e_1] \otimes_{\mathbb{F}} [e_2]) = (gx, [ge_1]_{gx} \otimes_{\mathbb{F}} [ge_2]_{gx}).$$

We then have the following result

Proposition 15. $\mathcal{E}_1 \otimes_k \mathcal{E}_2 \approx \Gamma(B_1 \otimes_{\mathbb{F}} B_2)$.

Proof. Define a map $\tilde{\phi} : \mathcal{E}_1 \times \mathcal{E}_2 \longrightarrow \Gamma(B_1 \otimes_{\mathbb{F}} B_2)$ by

$$\tilde{\phi}(s_1, s_2)(x) = s_1(x) \otimes_{\mathbb{F}} s_2(x).$$

We have

$$\begin{aligned}
\tilde{\phi}(s_1 + s_1, s_2) &= (s_1 + s_1)(x) \otimes_{\mathbb{F}} s_2(x) = (s_1(x) + s_1(x)) \otimes_{\mathbb{F}} s_2(x) \\
&= s_1(x) \otimes_{\mathbb{F}} s_2(x) + s_1(x) \otimes_{\mathbb{F}} s_2(x) = (\phi(s_1, s_2) + \phi(s_1, s_2))(x), \\
\tilde{\phi}(fs_1, s_2)(x) &= (fs_1)(x) \otimes_{\mathbb{F}} s_2(x) = (f(x)s_1(x)) \otimes_{\mathbb{F}} s_2(x) \\
&= s_1(x) \otimes_{\mathbb{F}} (f(x)s_2(x)) = s_1(x) \otimes_{\mathbb{F}} (fs_2)(x) = \phi(s_1, fs_2)(x).
\end{aligned}$$

So $\tilde{\phi}$ is k-bilinear and therefore induces a unique map $\phi : \mathcal{E}_1 \otimes_k \mathcal{E}_2 \longrightarrow \Gamma(B_1 \otimes_{\mathbb{F}} B_2)$ where

$$\phi(s_1 \otimes_k s_2)(x) = s_1(x) \otimes_{\mathbb{F}} s_2(x).$$

Let $\{s_i^1\}_{i=1}^n$ and $\{s_i^2\}_{i=1}^n$ be bases for $\mathcal{E}_1$ and $\mathcal{E}_2$ as k-modules. These bases exists because the modules are free as modules over k. Let $s \in \mathcal{E}_1 \otimes_k \mathcal{E}_2$ then

$$s = \sum_{ij} f_{ij} s_i^1 \otimes_k s_j^2.$$

Assume that $\phi(s) = 0$. This implies that

$$s = \sum_{ij} f_{ij}(x) s_i^1(x) \otimes_{\mathbb{F}} s_j^2(x) = 0 \quad \forall x \in \mathcal{S}.$$

But then $f_{ij}(x) = 0 \quad \forall x \in \mathcal{S}$ and so $f_{ij} = 0$ and as a consequence $s = 0$. So ϕ is injective. Let $s \in \Gamma(B_1 \otimes_{\mathbb{F}} B_2)$. This implies that $s(x) \in B_1 \otimes_{\mathbb{F}} B_2$ so there exists elements of $\mathbb{F}$ z_{ij}^x such that

$$s(x) = \sum_{ij} z_{ij}^x s_1(x) \otimes_{\mathbb{F}} s_2(x).$$

Define elements $f_{ij} \in$ k by $f_{ij}(x) = z^x_{ij}$ and define $h \in \mathcal{E}_1 \otimes_k \mathcal{E}_2$ by

$$h = \sum_{ij} f_{ij} s^1_i \otimes_k s^2_j.$$

Then we evidently have $\phi(h) = s$ and ϕ is surjective.

We already know that ϕ is k-linear by definition of tensor product. But we also have

$$\begin{aligned}\phi(g(s_1 \otimes_k s_2))(x) &= \phi(gs_1 \otimes_k gs_2)(x) = (gs_1)(x) \otimes_{\mathbb{F}} (gs_2)(x)\\ &= g(s_1(g^{-1}x)) \otimes_{\mathbb{F}} g(s_2(g^{-1}x)) = g((s_1 \otimes_k s_2)(g^{-1}x))\\ &= g(\phi(s_1, s_2)(g^{-1}x)) = (g\phi(s_1 \otimes_k s_2))(x)\end{aligned}$$

so ϕ is a A-module morphism and the proof is complete □

Now let $\mathcal{E} \approx \Gamma(B)$ be a GF-difference equation with corresponding vectorbundle B. Let $\mathcal{E}' \subset \mathcal{E}$ be a subequation. Define

$$V_x = \{[e]_x \mid e \in \mathcal{E}'\}.$$

Then $V_x \subset E_x$ is a subspace of E_x for each $x \in \mathcal{S}$ and the dimension is independent of x. Define a vectorbundle B' by

$$B' = \bigcup_{x \in \mathcal{S}} (x, V_x).$$

Then B' is evidently a subvectorbundle of B and we have by construction that $\mathcal{E}' \approx \Gamma(B')$. Let E_x/V_x be the factor space. Its dimension is independent of x and we can form the vectorbundle

$$B/B' = \bigcup_{x \in \mathcal{S}} (x, E_x/V_x).$$

Denote the elements of E_x/V_x by $[v_x]_{V_x}$. We define an action by elements in G by

$$g([v_x]_{V_x}) = [g(v_x)]_{V_{gx}}.$$

This action is well defined and we use it to induce an action of G on $\Gamma(B/B')$ in the usual way.

Proposition 16. $\mathcal{E}/\mathcal{E}' \approx \Gamma(B/B')$

Proof. Define a map $\phi : \Gamma(B)/\Gamma(B') \longrightarrow \Gamma(B/B')$ by $\phi([s])(x) = [s(x)]_{V_x}$. Then ϕ is well defined because if $[s] = [s']$ then $s - s' \in \Gamma(B')$ and therefore $s(x) - s'(x) \in V_x$. So $[s(x)]_{V_x} = [s'(x)]_{V_x}$ and therefore $\phi([s]) = \phi([s'])$. Furthermore we have

$$\begin{aligned}\phi([s] + [s'])(x) &= \phi([s + s'])(x) = [(s + s')(x)]_{V_x}\\ &= [s(x) + s'(x)]_{V_x} = [s(x)]_{V_x} + [s'(x)]_{V_x} = (\phi([s]) + \phi([s']))(x),\\ \phi(f[s])(x) &= \phi([fs])(x) = [(fs)(x)]_{V_x} = [f(x)s(x)]_{V_x}\\ &= f(x)[s(x)]_{V_x} = f(x)\phi([s])(x) = (f\phi([s]))(x),\\ \phi(g[s])(x) &= \phi([gs])(x) = [(gs)(x)]_{V_x} = [g(s(g^{-1}x))]_{V_x}\\ &= [g(s(g^{-1}x))]_{V_{g(g^{-1}x)}} = g([s(g^{-1}x)]_{V_{g^{-1}x}})\\ &= g(\phi([s])(g^{-1}x) = (g\phi([s]))(x).\end{aligned}$$

So we can conclude that ϕ is a A-module morphism. Assume that $\phi([s]) = \phi([s'])$. Then $\phi([s])(x) = \phi([s'])(x)$ for all $x \in \mathcal{S}$. But this is the same as $[s(x)]_{V_x} = [s'(x)]_{V_x}$ so $s(x) - s'(x) \in V_x$. This implies that $s - s' \in \Gamma(B')$ so by definition $[s] = [s']$ and ϕ is injective. Let $\gamma \in \Gamma(B/B')$, then $\gamma(x) = [v_x]_{V_x}$. Define $s \in \Gamma(B)$ by $s(x) = v_x$. Then clearly $\phi([s])(y) = [s(y)]_{V_y}$ so $\phi([s]) = \gamma$ and ϕ is surjective. □

Since $\wedge^n \mathcal{E}$ and $S^n \mathcal{E}$ are factor bundles of the tensor algebra $T\mathcal{E}$, it follows from the previous proposition that

Corollary 5.

$$\wedge^n \Gamma(B) = \Gamma(\wedge^n B),$$
$$S^n \Gamma(B) = \Gamma(S^n B).$$

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be GF-difference equations. We have proved that $Hom_k(\mathcal{E}_1, \mathcal{E}_2)$ is a GF-difference equation with the actions

$$(f\phi)(s_1) = f(\phi(s_1)) \quad \forall f \in k,$$
$$(g\phi)(s_1) = g(\phi(g^{-1}s_1)).$$

We know that $\mathcal{E}_1 \approx \Gamma(B_1)$, $\mathcal{E}_2 \approx \Gamma(B_2)$ where $B_1 = \bigcup_{x\in\mathcal{S}}(x, E^1_x)$ and $B_2 = \bigcup_{x\in\mathcal{S}}(x, E^2_x)$ are vectorbundles. Let $Hom_{\mathbb{F}}(E^1_x, E^2_x)$ be the set of $\mathbb{F}$-linear maps from E^1_x to E^2_x. These have all the same dimension and we can form the vector bundle

$$Hom_{\mathbb{F}}(B_1, B_2) = \bigcup_{x\in\mathcal{S}} (x, Hom_{\mathbb{F}}(E^1_x, E^2_x))$$

We have a G-action on the vectorbundle $Hom_{\mathbb{F}}(B_1, B_2)$ given by $g(x, \phi_x) = (gx, g(\phi_x))$ where we define

$$g(\phi_x)([e_1]_{gx}) = g(\phi_x(g^{-1}[e_1]_{gx}))$$

This induces the structure of a left A-module on the set $\Gamma(Hom_{\mathbb{F}}(B_1, B_2))$ in the usual way.

Proposition 17. $Hom_k(\mathcal{E}_1, \mathcal{E}_2) \approx \Gamma(Hom_{\mathbb{F}}(B_1, B_2))$

Proof. Define a map

$$F : Hom_k(\mathcal{E}_1, \mathcal{E}_2) \longrightarrow \Gamma(Hom_{\mathbb{F}}(B_1, B_2)),$$
$$\phi \longmapsto F\phi$$

as $(F\phi)(x)(v^1_x) = \phi(s)(x)$ where $s \in \Gamma(B_1)$ satisfies $s(x) = v^1_x$. This is well defined because if $s, s' \in \Gamma(B_1)$ and $s(x) = s'(x)$ for the given $x \in \mathcal{S}$ then

$$\phi(s)(x) = (\delta_x\phi(s))(x) = (\phi(\delta_x s))(x).$$
$$\phi(s')(x) = (\delta_x\phi(s'))(x) = (\phi(\delta_x s'))(x).$$

But

$$(\delta_x s)(y) = \delta_x(y)s(y) = \delta_{xy}v^1_x,$$
$$(\delta_x s')(y) = \delta_x(y)s'(y) = \delta_{xy}v^1_x.$$

Therefore $\delta_x s = \delta_x s'$ and we have $\phi(s)(x) = (\phi(\delta_x s))(x) = (\phi(\delta_x s'))(x) = \phi(s')(x)$. So F is well defined. Assume that $F\phi_1 = F\phi_2$. Let $s \in \Gamma(B_1)$, then $\phi_1(s)(x) = (F\phi_1)(x)(s(x)) = (F\phi_2)(x)(s(x)) = \phi_2(s)(x)$ for all $x \in \mathcal{S}$. But then $\phi_1 = \phi_2$ and F

is injective. Let $\gamma \in \Gamma(Hom_{\mathbb{F}}(B_1, B_2))$ be given. Define a map $\phi : \Gamma(B_1) \longrightarrow \Gamma(B_2)$ by $\phi(s)(x) = \gamma(x)(s(x))$. Then we have

$$\begin{aligned}\phi(s+s')(x) &= \gamma(x)(s(x)+s'(x)) \\ &= \gamma(x)(s(x)) + \gamma(x)(s'(x)) = \phi(s)(x) + \phi(s')(x), \\ \phi(fs)(x) &= \gamma(x)(f(x)s(x)) = f(x)\gamma(x)(s(x)) \\ &= f(x)\phi(s)(x) = (f\phi(s))(x).\end{aligned}$$

So we have that $\phi \in Hom_k(\Gamma(B_1), \Gamma(B_2))$ and also

$$F(\phi)(x)(v_x^1) = \phi(s)(x) = \gamma(x)(s(x)) = \gamma(x)(v_x^1).$$

Therefore we have that $F\phi = \gamma$ and F is surjective. Furthermore we have

$$\begin{aligned}F(f\phi)(x)(v_x^1) &= (f\phi)(s)(x) = (f(\phi(s)))(x) \\ &= f(x)(\phi(s)(x)) = f(x)(F(\phi)(x)(v_x^1)) = (f(x)F(\phi)(x))(v_x^1) = (fF(\phi))(x)(v_x^1), \\ F(g\phi)(x)(v_x^1) &= (g\phi)(s)(x) = (g(\phi(g^{-1}s)))(x) = g(\phi(g^{-1}s)(g^{-1}x)) \\ &= g(F(\phi)(g^{-1}x)(g^{-1}v_x^1)) = (g(F(\phi)(g^{-1}x)))(v_x^1) \\ &= (gF(\phi))(x)(v_x^1).\end{aligned}$$

So F is a A-module morphism. $\square$

Let $\mathcal{E} \approx \Gamma(B)$ be a given GF-difference equation where $B = \bigcup_{x\in\mathcal{S}}(x, E_x)$ is a vectorbundle. Define the dual vector bundle $B^* = \bigcup_{x\in\mathcal{S}}(x, E_x^*)$. Then as a special case of the previous proposition we have.

Proposition 18. $\mathcal{E}^* \approx \Gamma(B^*)$.

3.3. **The geometric description of A-morphisms.** Let $\Gamma(B), \Gamma(B')$ be two GF-difference equations with corresponding vectorbundles $B = \bigcup_{x\in\mathcal{S}}(x, E_x)$, $B' = \bigcup_{x\in\mathcal{S}}(x, E'_x)$ and let $\phi \in Hom_A(\Gamma(B), \Gamma(B'))$. Define a map $F_\phi : B \longrightarrow B'$ by

$$F_\phi(x, v_x) = (x, F_\phi^x)$$

where $F_\phi^x(v_x) = \phi(s)(x)$ and $s \in \Gamma(B)$ is any section satisfying $s(x) = v_x$.

Proposition 19. *$F_\phi^x : E_x \longrightarrow E'_x$ is well defined.*

Proof. Assume $s(x) = s'(x) = v_x$. then

$$\begin{aligned}\phi(s)(x) &= (\delta_x\phi(s))(x) = \phi(\delta_x s)(x), \\ \phi(s')(x) &= (\delta_x\phi(s'))(x) = \phi(\delta_x s')(x),\end{aligned}$$

and $(\delta_x s)(y) = \delta_x(y)s(y) = \delta_x(y)s'(y) = (\delta_x s')(y)$ for all $y \in \mathcal{S}$. This means that $\delta_x s = \delta_x s'$ and so $\phi(\delta_x s) = \phi(\delta_x s')$ and we can conclude that $\phi(s)(x) = \phi(s')(x)$. $\square$

Proposition 20. *F_ϕ^x is $\mathbb{F}$-linear.*

Proof. Let $v_x, u_x \in E_x$ and let $s, t \in \Gamma(B)$ be any sections such that $s(x) = v_x, t(x) = u_x$. Then $(s+t)(x) = v_x + u_x$ and we have

$$F_\phi^x(v_x + u_x) = \phi(s+t)(x) = \phi(s)(x) + \phi(t)(x) = F_\phi^x(v_x) + F_\phi^x(u_x).$$

Let $a \in \mathbb{F}$ and $v_x \in E_x$. Let $s \in \Gamma(B)$ be any section such that $s(x) = v_x$. Then $(as)(x) = a(s(x)) = av_x$ and we have

$$F^x_\phi(av_x) = \phi(as)(x) = (a\phi(s))(x) = a(\phi(s)(x)) = aF^x_\phi(v_x),$$

□

Proposition 21. *$F^y_\phi \circ g = g \circ F^{g^{-1}y}_\phi$ for all $g \in G$ and $y \in \mathcal{S}$.*

Proof. Let $y \in \mathcal{S}, g \in G$ and $v_{g^{-1}y} \in E_{g^{-1}y}$. Let $s \in \Gamma(B)$ be any section such that $s(g^{-1}y) = v_{g^{-1}y}$. Then we have

$$\begin{aligned} F^y_\phi(gv_{g^{-1}y}) &= F^y_\phi(gs(g^{-1}y)) = F^y_\phi((gs)(y)) = \phi(gs)(y) \\ &= (g\phi(s))(y) = g(\phi(s)(g^{-1}y)) = g(F^{g^{-1}y}_\phi(s(g^{-1}y))) = g(F^{g^{-1}y}_\phi(v_{g^{-1}y}). \end{aligned}$$

□

The previous three propositions show that a morphism of GF-difference equations is a family of $\mathbb{F}$-linear maps that are related at different points as described in the last proposition. Let H_y be the isotropy group of the point $y \in \mathcal{S}$. As a special case of the last proposition we have

Corollary 6. *$F^y_\phi \circ h = h \circ F^y_\phi$ for all $h \in H_y$ and $y \in \mathcal{S}$.*

So the maps F^y_ϕ commutes with the action of the isotropy group at each point and are $\mathbb{F}[H_x]$-module morphisms on the fibre above the point. Properties of the morphisms $\phi \in Hom(\Gamma(B), \Gamma(B'))$ is transferred to the family of maps F^x_ϕ.

Proposition 22. *Let $x \in \mathcal{S}$ be some point in $\mathcal{S}$. Then $\phi \in Hom_A(\Gamma(B), \Gamma(B'))$ is surjective if and only if $F^x_\phi : E_x \longrightarrow E'_x$ is surjective.*

Proof. Assume ϕ is surjective. Let $v'_x \in E'_x$ be given. Then there exists $\gamma \in \Gamma(B')$ such that $\gamma(x) = v'_x$. Let $s \in \Gamma(B)$ be such that $\phi(s) = \gamma$. Let $v_x = s(x)$. Then

$$F^x_\phi(v_x) = \phi(s)(x) = \gamma(x) = v'_x$$

so F^x_ϕ is surjective.

Assume that F^x_ϕ is surjective. Let $y \in \mathcal{S}$ and let $v'_y \in E'_y$. There exists $g \in G$ such that $gx = y$. Define $v'_x = g^{-1}v'_y \in E'_x$. Then there exists $v_x \in E_x$ such that $F^x_\phi(v_x) = v'_x$. Define $v_y = gv_x$. Then we have

$$F^y_\phi(v_y) = F^y_\phi(gv_x) = gF^x_\phi(v_x) = gv'_x = v'_y$$

so F^y_ϕ is surjective for all $y \in \mathcal{S}$. Let $\gamma \in \Gamma(B')$. Then $\gamma(y) = v_y \in E'_y$ for all y. For each y there then exists $v_y \in E_y$ such that $F^y_\phi(v_y) = v'_y$. Define $s \in \Gamma(B)$ by $s(y) = v_y$. Then we have

$$\phi(s)(y) = F^y_\phi(v_y) = v'_y = \gamma(y)$$

so $\phi(s) = \gamma$ and ϕ is surjective. □

Proposition 23. *Let $x \in \mathcal{S}$ be any point in $\mathcal{S}$. Then $\phi \in Hom_A(\Gamma(B), \Gamma(B'))$ is injective if and only if $F^x_\phi : E_x \longrightarrow E'_x$ is injective.*

Proof. Assume that $\phi \in Hom_A(\Gamma(B), \Gamma(B'))$ is not injective. Then there exists $s \in \Gamma(B)$,$s \neq 0$ such that $\phi(s) = 0$. There is at least one point $y \in \mathcal{S}$ such that $s(y) = v_y \neq 0$. Then

$$F^y_\phi(v_y) = \phi(s)(y) = 0$$

so F^y_ϕ is not injective. Let $g \in G$ be such that $gx = y$. Let $g : E_x \longrightarrow E_y$ be the corresponding invertible fibermap. Define $v_x = g^{-1}v_y$. Then $v_x \neq 0$,$v_x \in E_x$ and

$$F^x_\phi(v_x) = F^x_\phi(g^{-1}v_y) = g^{-1}F^y_\phi(v_y) = 0$$

so F^x_ϕ is not injective.

Assume that ϕ is injective. Let $v_x \in E_x$ and assume that $F^x_\phi(v_x) = 0$. Let $s \in \Gamma(B)$ be any section such that $s(x) = v_x$. Define $\gamma \in \Gamma(B)$ by $\gamma = \delta_x s$. Then

$$\phi(\gamma)(y) = \phi(\delta_x s)(y) = \delta_x(y)\phi(s)(y) = \delta_{xy}F^x_\phi(v_x) = 0$$

for all $y \in \mathcal{S}$. But then $\gamma = 0$ and so $v_x = s(x) = \gamma(x) = 0$ and we conclude that F^x_ϕ is injective. □

Combining the previous propositions we have

Corollary 7. *Let $x \in \mathcal{S}$ be any point in $\mathcal{S}$. Then $\phi \in Hom_A(\Gamma(B), \Gamma(B'))$ is an A-module isomorphism if and only if $F^x_\phi : E_x \longrightarrow E'_x$ is a $\mathbb{F}[H]$-module isomorphism.*

Any A-morphism gives us a family of $\mathbb{F}$-linear maps with the properties described. Any such family will in fact come from a A-morphism of modules.

For each $x \in \mathcal{S}$ let $F^x : E_x \longrightarrow E'_x$ be a $\mathbb{F}$-linear map. Assume that the members of the family are related through

$$F^x_\phi \circ g = g \circ F^{g^{-1}x}_\phi$$

for all $g \in G$ and $x \in \mathcal{S}$.

Define a map $\phi : \Gamma(B) \longrightarrow \Gamma(B')$ by

$$\phi(s)(x) = F^x(s(x)).$$

Proposition 24. $\phi \in Hom_A(\Gamma(B), \Gamma(B'))$

Proof. Let $s, t \in \Gamma(B)$. Then

$$\begin{aligned}\phi(s+t)(x) &= F^x((s+t)(x)) = F^x(s(x) + t(x)) \\ &= F^x(s(x)) + F^x(t(x)) = \phi(s)(x) + \phi(t)(x).\end{aligned}$$

Let $s \in \Gamma(B)$ and $f \in k$. Then

$$\begin{aligned}\phi(fs)(x) &= F^x((fs)(x)) = F^x(f(x)s(x)) \\ &= f(x)F^x(s(x)) = f(x)\phi(s)(x) = (f\phi(s))(x).\end{aligned}$$

Let $s \in \Gamma(B)$ and $g \in G$. Then

$$\begin{aligned}\phi(gs)(x) &= F^x((gs)(x)) = F^x(g(s(g^{-1}x))) \\ &= g(F^{g^{-1}x}(s(g^{-1}x))) = g(\phi(s)(g^{-1}x)) = (g\phi(s))(x)\end{aligned}$$

□

In general a submodule of a finitely generated module does not have to be finitely generated. We will now show that for the category $\mathbb{GFE}$ all submodules are in fact GF-difference equations.

Proposition 25. *Let $\mathcal{E}$ and $\mathcal{E}'$ be GF-difference equations and let $\phi \in Hom_A(\mathcal{E}, \mathcal{E}')$ be a A-module morphism. The $Im\phi$ and $Ker\phi$ are GF-difference equations.*

Proof. We know that $im\phi$ and $ker\phi$ are submodules of geometric modules and are therefore geometric. Let $\{e_i\}_{i=1}^n$ be a set of generators for $\mathcal{E}$. Then $\{\phi(e_i)\}_{i=1}^n$ is a finite set of generators for $im\phi$. So $im\phi$ is a GF-difference equation. We know that $\mathcal{E} \approx \Gamma(B)$ for some vectorbundle $B = \bigcup_{y \in \mathcal{S}}(y, E_y)$. We know that $ker\phi$ is a geometric submodule so we have an injective A-module morphism $T : ker\phi \hookrightarrow \Gamma(B') \subset \Gamma(B)$ where $B' = \bigcup_{y \in \mathcal{S}}(y, V_y)$ is the subbundle with fibers

$$V_y = \{s(y) \mid s \in ker\phi\}.$$

Let $\{F_\phi^y\}_{y \in \mathcal{S}}$ be the family of maps corresponding to ϕ and let $v_y \in V_y$. Then $v_y = s(y)$ for some $s \in ker\phi$ and we have $F_\phi^y(v_y) = \phi(s)(y) = 0$. Let $\gamma \in \Gamma(B')$. Then $\gamma(y) \in V_y$ for all $y \in \mathcal{S}$ so that $\phi(\gamma)(y) = F_\phi^y(\gamma(y)) = 0$ and as a concequence $\gamma \in ker\phi$ and the map T is surjective. But then we have proved that $ker\phi \approx \Gamma(B')$ and $ker\phi$ is finitely generated and therefore a GF-difference equation. □

Corollary 8. *Let $\mathcal{E}$ be a GF-difference equation and $\mathcal{E}' \subset \mathcal{E}$ a submodule. Then $\mathcal{E}'$ is a GF-difference equation.*

Proof. We know that $\mathcal{E}$ and $\mathcal{E}/\mathcal{E}'$ are GF-difference equations. Let $\phi : \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{E}'$ be the natural projection. Then ϕ is a A-module morphism and $\mathcal{E}' = ker\phi$. □

3.4. **The general structure of $\mathbb{GFE}$.** We are now ready to give a characterization of the structure of the category $\mathbb{GFE}$. All the structural properties follow from the following proposition.

Proposition 26. *Let $\mathcal{E}$ be a GF-difference equation. Then $\mathcal{E}$ is both Artinian and Noetherian.*

Proof. Let $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3 \cdots$ be a ascending chain of submodules in $\mathcal{E}$. Then in particular this is a chain of free k-modules. But $\mathcal{E}$ has finite dimension over k so the chain must stop and the module $\mathcal{E}$ is Noetherian. Similarly let $\cdots \subset \mathcal{E}_3 \subset \mathcal{E}_2 \subset \mathcal{E}_1$ be a decending chain of submodules of $\mathcal{E}$. Then in particular it is a decending chain of free k-modules. But the dimension of any module over k is nonnegative so the chain must stop. □

A GF-difference equation is *simple* if it contains no GF-difference equations as submodules and *indecomposable* if it can not be written as a direct sum of GF-difference equations. A composition series for a GF-difference equation is a finite filtering $0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \cdots \mathcal{E}$ of the equation $\mathcal{E}$ such that the composition factors $\mathcal{E}_i/\mathcal{E}_{i-1}$ are simple equations. Because of the previous proposition and the general structure theory for modules [?] we have

Theorem 1. *Let $\mathcal{E}$ be a GF-difference equation. Then $\mathcal{E}$ has a composition series and all composition series for $\mathcal{E}$ has the same number of elements in the filtration and the composition factors are the same up to isomorphism. Any GF-difference equation can be written as a direct sum of a finite number of indecomposable equations.*

This theorem is a combination of the Jordan-Hølder theorem and the Krull-Smidth teorem. This Theorem reduce the problem of solving GF-difference equations to the study of indecomposable equations. Furthermore it show that all indecomposable equations are related to simple equations through a finite set of simple decomposition factors. The first problem then is to find the simple equations and the next is to construct the indecomposable equations using a set of composition factors. This last problem is essentially the problem of quantization. Even finding the simple equations is in general not a trivial task in the category $\mathbb{GFE}$. We will however now proceed to prove a Theorem that show that the Category $\mathbb{GFE}$ is equivalent to a category where the problem of finding simple and indecomposable objects is more approachable.

4. The equivalence Theorem

We will first construct a special class of GF-difference equations and then show that all GF-difference equations are in fact of this type. Let $x \in \mathcal{S}$ and let H_x be the isotropy group of that point. All such groups for different points x are isomorphic. We will usually supress the point we are referring to and just write $H = H_x$. For each $y \in \mathcal{S}$ define the set yH by

$$yH = \{g \in G \mid gx = y\}.$$

We evidently have $gH = yH$ where $g \in G$ satisfy $gx = y$ so the sets yH are just the left cosets of H in G. Let V be a finite dimensional $\mathbb{F}[H]$-module and form the trivial bundle $\mathcal{S} \times V$

$$\mathcal{S} \times V = \bigcup_{y \in \mathcal{S}} (y, V),$$

Let σ be a transversal to the partitioning of G by the classes yH

$$\sigma(y) \in yH \quad \forall y \in \mathcal{S},$$

Let $\Gamma(G \times V)$ be the k-module of sections in the trivial bundle $\mathcal{S} \times V$. We will now define an action of G in this module of sections as

$$g(y, v) = (gy, \sigma(gy)^{-1} g\sigma(y)v).$$

This gives a G-action.

Proposition 27. $(gg')(y, v) = g(g'(y, v))$

Proof.

$$\begin{aligned}
(gg')(y, v) &= ((gg')y, \sigma((gg')y)(gg')\sigma(y)v) \\
&= (g(g'y), \sigma(g(g'y))g\sigma(g'y)\sigma(g'y)^{-1}g'\sigma(y)v) \\
&= g(g'y, \sigma(g'y)g'\sigma(y)v) = g(g'(y, v)).
\end{aligned}$$

□

We let this action in the bundle induce an action on the module of sections in the bundle in the usual way.

$$(gs)(y) = g(s(g^{-1}y))$$

This gives the set of section in the bundle $\mathcal{S} \times V$ the structure of a GF-difference equation. It appears as if each choise of transversal σ gives a different action on the bundle and so a different A-module structure on the set $\Gamma(\mathcal{S} \times V)$. They are however all isomorphic. Let $\Gamma(\mathcal{S} \times V)$ and $\Gamma(\mathcal{S} \times V)'$ be the modules corresponding to the choise of two transversals σ and σ'. Then we have

Proposition 28. $\Gamma(\mathcal{S} \times V) \approx \Gamma(\mathcal{S} \times V)'$

Proof. For each $y \in \mathcal{S}$ there exists a $\gamma(y) \in H$ such that $\sigma(y) = \sigma'(y)\gamma(y)$. Define a map $\phi_\gamma : \Gamma(\mathcal{S} \times V) \longrightarrow \Gamma(\mathcal{S} \times V)'$ by $\phi_\gamma(s)(y) = \gamma(y)^{-1}(s(y))$ This map is clearly bijective with inverse $\phi_{\gamma^{-1}}$ where $\phi_{\gamma^{-1}}(s)(y) = \gamma(y)(s(y))$. We also have

$$\begin{aligned}
\phi_\gamma(fs)(y) &= \gamma(y)^{-1}((fs)(y)) = \gamma(y)^{-1}(f(y)s(y)) = f(y)(\gamma(y)^{-1}(s(y))) \\
&= f(y)\phi_\gamma(s)(y) = (f\phi_\gamma(s))(y), \\
\phi_\gamma(gs)(y) &= \gamma(y)^{-1}((gs)(y)) = \gamma(y)^{-1}(g(s(g^{-1}y))) \\
&= \gamma(y)^{-1}(\sigma(y)^{-1}g\sigma(g^{-1}y)s(g^{-1}y)) \\
&= \gamma(y)^{-1}\sigma(y)^{-1}g\sigma(g^{-1}y)\gamma(g^{-1}y)\gamma(g^{-1}y)^{-1}s(g^{-1}y) \\
&= (\sigma(y)\gamma(y))^{-1}g(\sigma(g^{-1}y)\gamma(g^{-1}y))\gamma(g^{-1}y)^{-1}s(g^{-1}y) \\
&= \sigma'(y)^{-1}g\sigma'(g^{-1}y)(\phi_\gamma(s))(g^{-1}y) \\
&= (g\phi_\gamma(s))(y).
\end{aligned}$$

So the map ϕ_γ is also a A-module morphism and the proof is complete. □

The constructed class of GF-difference equations in fact includes all GF-difference equations.

Theorem 2. *Let $\mathcal{E}$ be any GF-difference equation. Then $\mathcal{E} \approx \Gamma(\mathcal{S} \times V)$ for some $\mathbb{F}[H]$-module of finite dimension over $\mathbb{F}$.*

Proof. We know that $\mathcal{E} \approx \Gamma(B)$ for some vectorbundle B

$$B = \bigcup_{y \in \mathcal{S}} (y, E_y).$$

Let σ be a transversal to the classes yH, Then $\sigma(y)x = y$ and so the action of G on the bundle gives us the lift $\sigma(y) : E_x \longrightarrow E_y$ and this map is an isomorphism since G is a group. Define $V = E_x$. Then V is an finite dimensional H space. Let ϕ be the map

$$\begin{aligned}
\phi : B &\longrightarrow \mathcal{S} \times V \\
(y, v_y) &\longmapsto (y, \sigma(y)^{-1}v_y).
\end{aligned}$$

This map is clearly an isomorphism of vectorbundles and it commutes with the G action

$$\begin{aligned}
\phi(g(y, v_y)) &= \phi(gy, gv_y) = (gy, \sigma(gy)^{-1}gv_y) \\
&= (gy, \sigma(gy)^{-1}g\sigma(y)\sigma(y)^{-1}v_y) = g\phi(y, v_y).
\end{aligned}$$

Define a map F_ϕ by

$$\begin{aligned}
F_\phi : \Gamma(B) &\longrightarrow \Gamma(\mathcal{S} \times V) \\
F_\phi(s)(y) &\longmapsto \phi(s(y))
\end{aligned}$$

F_ϕ is clearly bijective with inverse $F_{\phi^{-1}}$ and we also have

$$\begin{aligned}
F_\phi(fs)(y) &= \phi((fs)(y)) = \phi(f(y)s(y)) = f(y)\phi(s(y)) \\
&= f(y)(F_\phi(s)(y)) = (fF_\phi(s))(y), \\
F_\phi(gs)(y) &= \phi((gs)(y)) = \phi(g(s(g^{-1}y))) \\
&= g(\phi(s(g^{-1}y))) = g(F_\phi(s)(g^{-1}y)) = (gF_\phi(s))(y).
\end{aligned}$$

So F_ϕ is a A-module morphism and the proof is complete. □

Let $\mathbb{F}[H]-finmod$ be the category of modules over $\mathbb{F}[H]$ with finite dimension over $\mathbb{F}$ with direct sum and tensor product and dual over $\mathbb{F}$ defined as is usual in representation theory. For this category all main structural theorems for the decomposition of modules apply so that all such modules have compositionseries and can be written as a finite direct sum of indecomposables. We will now proceed to show that the category $\mathbb{GFE}$ and $\mathbb{F}[H]-finmod$ are in fact equivalent as categories. From a structural point of view we will not distinguish between isomorphic objects and will therefore prove the equivalence by showing the isomorphism of the Grotendick algebra [6] A_G for $\mathbb{GFE}$ and A_H for $\mathbb{F}[H]-finmod$. The algebra structure in A_G and A_H is the one induced from direct sum and tensor product in the corresponding categories. In addition to the usual algebra structure we have a conjugation map induced from the dual in the categories. Define a map on $T : A_H \longrightarrow A_G$ by

$$T([V]) = [\Gamma(\mathcal{S} \times V)]$$

where elements in the Grotendick algebras are denoted by square brackets of elements in the corresponding categories.

Proposition 29. *T is well defined*

Proof. Assume $[V] = [U]$. Then there exists a $\mathbb{F}[H]$-module isomorphism $F^x : V \longrightarrow U$. So F^x is an $\mathbb{F}$-isomorphism and $F^x(hv) = hF^x(v)$ for all $h \in H$. Denote the fiber over $y \in \mathcal{S}$ of the vectorbundles $B_1 = \mathcal{S} \times V$ and $B_2 = \mathcal{S} \times U$ by V_y and U_y. For each $y \in \mathcal{S}$ define a map $F^y : V_y \longrightarrow U_y$ by $F^y(v_y) = g(F^x(g^{-1}v_y)) \in U_y$ where g is any element in G such that $gx = y$. The family of maps $\{F^y\}_{y\in\mathcal{S}}$ is well defined because if g_1 is another element in G such that $g_1x = y$ then $g_1 = gh$ for some $h \in H$ and we have

$$g_1(F^x(g_1^{-1}v_y)) = (gh)(F^x((gh)^{-1}v_y)) = g(h(F^x(h^{-1}g^{-1}v_y))) = g(F^x(g^{-1}v_y)).$$

The family $\{F^y\}_{y\in\mathcal{S}}$ satisfies all requirements in proposition (24) and therefore determines a A-morphism, $\phi : \Gamma(\mathcal{S} \times V) \longrightarrow \Gamma(\mathcal{S} \times U)$. From the construction we observe that each member of the family $\{F^y\}_{y\in\mathcal{S}}$ is a $\mathbb{F}[H]$-module isomorphism. We therefore can conclude that the map ϕ is an A-isomorphism so that $[\Gamma(\mathcal{S}\times V)] = [\Gamma(\mathcal{S} \times U)]$. ◻

Proposition 30. *$T : A_H \longrightarrow A_G$ is a bijection.*

Proof. Let $[\mathcal{E}]$ be any element in A_G. From theorem 2 we know that there exists a V in $\mathbb{F}[H]-finmod$ such that $[\mathcal{E}] = [\Gamma(\mathcal{S} \times V)]$. The $T([V]) = [\mathcal{E}]$ so that T is surjective. Assume that $T([V]) = T([U])$. This means that there exists a A-module isomorphism $\phi : \Gamma(\mathcal{S} \times V) \longrightarrow \Gamma(\mathcal{S} \times U)$. Let $y \in \mathcal{S}$ be any point.Then proposition7 show that there exists a $\mathbb{F}[H]$-module isomorphism $F^y_\phi : V \longrightarrow U$. But then $[V] = [U]$ and T is injective. ◻

Rewriting some of the results proved earlier we find that T is a structure preserving map.

Proposition 31. *The map T is structure preserving*

$$T([U] + [V]) = T([U]) + T([V])$$
$$T([U][V]) = T([U])T([V])$$
$$T([U]^*) = (T([U])^*$$

This relation between the categories $\mathbb{GFE}$ and $\mathbb{F}[H]-finmod$ gives us a way to find all indecomposable equations of finite type in $\mathbb{GFE}$ from the indecomposable $\mathbb{F}[H]$-modules of finite dimension over $\mathbb{F}$.

Proposition 32. *$\mathcal{E}$ is a indecomposable GF-difference equation of finite type if and only if $\mathcal{E} \approx \Gamma(\mathcal{S} \times V)$ where is a indecomposable $\mathbb{F}[H]$-module of finite dimension over $\mathbb{F}$.*

Proof. Let $\mathcal{E}$ be indecomposable. We know that $\mathcal{E} \approx \Gamma(\mathcal{S} \times V)$ for some $\mathbb{F}[H]$-module V. Assume that V is decomposable so that $V \approx V_1 \oplus V_2$. Define $\mathcal{E}_1 = \Gamma(\mathcal{S} \times V_1), \mathcal{E}_2 = \Gamma(\mathcal{S} \times V_2)$. Then

$$\begin{aligned}[\mathcal{E}] &= T([V]) = T([V_1 \oplus V_2]) = T([V_1] + [V_2]) \\ &= T([V_1]) + T([V_2]) = [\mathcal{E}_1] + [\mathcal{E}_2] = [\mathcal{E}_1 \oplus \mathcal{E}_2].\end{aligned}$$

So that $\mathcal{E} \approx \mathcal{E}_1 \oplus \mathcal{E}_2$ and $\mathcal{E}$ is decomposable. This is a contradiction so that V is indecomposable. Let V be indecomposable and of finite dimension over $\mathbb{F}$. Define $\mathcal{E} = \Gamma(\mathcal{S} \times V)$. Then $\mathcal{E}$ is a GF-difference equation of finite type and $T([V]) = [\mathcal{E}]$. Assume that $\mathcal{E}$ is decomposable. Then $\mathcal{E} \approx \mathcal{E}_1 \oplus \mathcal{E}_2$. We know that there exists $\mathbb{F}[H]$-modules V_1, V_2 of finite dimension over $\mathbb{F}$ such that $\mathcal{E}_1 \approx \Gamma(\mathcal{S} \times V_1), \mathcal{E}_2 \approx \Gamma(\mathcal{S} \times V_2)$. Then we have

$$\begin{aligned}T[V] &= [\mathcal{E}] = [\mathcal{E}_1 \oplus \mathcal{E}_2] = [\mathcal{E}_1] + [\mathcal{E}_2] = T([V_1]) + T([V_2]) \\ &= T([V_1] + [V_2]) = T([V_1 \oplus V_2]).\end{aligned}$$

But T is injective so that $[V] = [V_1 \oplus V_2]$. Then $V \approx V_1 \oplus V_2$ and V is decomposable. This is a contradiction. □

For simple equations of finite type we have

Proposition 33. *$\mathcal{E}$ is a simple GF-difference equation of finite type if and only if $\mathcal{E} \approx \Gamma(\mathcal{S} \times V)$ where V is a simple $\mathbb{F}[H]$-module.*

Proof. Let $\mathcal{E}$ be a simple GF-difference equation of finite type. We know that $\mathcal{E} \approx \Gamma(\mathcal{S} \times V)$ for some $\mathbb{F}[H]$-module V of finite dimension over $\mathbb{F}$. Assume that V has a submodule V'. let $\mathcal{E}' = \Gamma(\mathcal{S} \times V')$. Then $\mathcal{E}'$ is a submodule of $\mathcal{E}$ so $\mathcal{E}$ is not simple. This is a contradiction. Assume that V is a simple $\mathbb{F}[H]$-module. Let $\mathcal{E} = \Gamma(\mathcal{S} \times V)$ and assume that $\mathcal{E}$ is not simple so that it has a submodule $\mathcal{E}'$. But then $\mathcal{E}' \approx \Gamma(\mathcal{S} \times V')$ where V' is a submodule of V. This is a contradiction. □

The simple $\mathbb{F}[H]$-modules are in general not easy to find. For the case when the isotropy group is finite and the character of the field does not divide the order of the group H, the algebra $\mathbb{F}[H]$ is semisimple and the full power of the theory of characters [7] can be applied. Even in the case when the character of $\mathbb{F}$ does divide the order of the group, the modular case, powerful tools are available.

5. The projection formula for GF-difference equations

The Frobenius projection formula [8] can be generalized to apply to GF-difference equations when the isotropy group is finite and the underlying field is $\mathbb{C}$. This formula greatly simplifies the solution process when it applies. Let $\mathcal{E} \approx \Gamma(B)$ be any GF-difference equation where $B = \bigcup_{y \in \mathcal{S}} (y, E_y)$ is a vectorbundle over $\mathcal{S}$. We know that the fiber over $y \in \mathcal{S}$ is a $\mathbb{F}[H_y]$-module. Denote the character of this module by χ_y. There is a relation between characters at different points.

Proposition 34. *$\chi_y(h_y) = \chi_{gy}(gh_yg^{-1})$ for all $y \in \mathcal{S}, g \in G$ and $h_y \in H_y$.*

Proof. Let $h_y \in H_y$ and let $g \in G$. Then $h_y = g^{-1}gh_yg^{-1}g = g^{-1}(gh_yg^{-1})g$ so we have

$$\chi_y(h_y) = tr(h_y) = tr(g^{-1}(gh_yg^{-1})g) = tr(gh_yg^{-1}) = \chi_{gy}(gh_yg^{-1}).$$

□

Assume that $\mathcal{E}' \approx \Gamma(B')$ is a submodule of $\mathcal{E} \approx \Gamma(B)$. Then $B' \subset B$ so that $E'_y \subset E_y$ as a $\mathbb{F}$-linear subspace for all $y \in \mathcal{S}$. For each $y \in \mathcal{S}$ let $\Pi_{\mathcal{E}'}(y) : E_y \longrightarrow E_y$ be the Frobenius map

$$\Pi_{\mathcal{E}'}(y)v_y = \frac{dimE'_y}{\mid H_y \mid} \sum_{h_y \in H_y} \chi_y(h_y^{-1})h_yv_y.$$

Here χ_y is the character of the $\mathbb{F}[H]$-module E'_y.

Each h_y is a $\mathbb{F}$-linear map so clearly $\Pi_{\mathcal{E}'}(y)$ is a $\mathbb{F}$-linear map for each $y \in \mathcal{S}$. We also have

Proposition 35. *$\Pi_{\mathcal{E}'}(y)(gv_{g^{-1}y}) = g\Pi_{\mathcal{E}'}(g^{-1}y)(v_{g^{-1}y})$ for all $y \in \mathcal{S}$ and $g \in G$.*

Proof. Let $c_y = dimE_y / \mid H_y \mid$. Then $c_y = c_{y'}$ for all $y, y' \in \mathcal{S}$ since $B = \bigcup_{y\in\mathcal{S}}(y, E_y)$ is a vectorbundle and $H_y \approx H_{y'}$ for all $y, y' \in \mathcal{S}$. We have

$$\begin{aligned}
\Pi_{\mathcal{E}'}(y)(g^{-1}v_y) &= c_y \sum_{h_y\in H_y} \chi_y(h_y^{-1})h_y(gv_{g^{-1}y}) = c_y \sum_{h_y\in H_y} \chi_y(h_y^{-1})g(g^{-1}h_yg)v_{g^{-1}y}) \\
&= c_yg \sum_{h_y\in H_y} \chi_y(h_y^{-1})(g^{-1}h_yg)v_{g^{-1}y} \\
&= c_yg \sum_{h_{g^{-1}y}\in H_{g^{-1}y}} \chi_y(gh_y^{-1}g^{-1})h_{g^{-1}y}v_{g^{-1}y} \\
&= c_{g^{-1}y}g \sum_{h_{g^{-1}y}\in H_{g^{-1}y}} \chi_{g^{-1}y}(h_{g^{-1}y}^{-1})h_{g^{-1}y}v_{g^{-1}y} \\
&= \Pi_{\mathcal{E}'}(g^{-1}y)(v_{g^{-1}y}).
\end{aligned}$$

□

Define a map $\Pi_{\mathcal{E}'}$ on $\mathcal{E} \approx \Gamma(B)$ by

$$\Pi_{\mathcal{E}'}(e)(y) = \Pi_{\mathcal{E}'}(y)(e(y)).$$

Then we can conclude from the previous proposition that

Corollary 9. $\Pi_{\mathcal{E}'} \in Hom_A(\mathcal{E}, \mathcal{E})$.

In general $\Pi_{\mathcal{E}'}$ is not a projection on $\mathcal{E}'$. Assume that $\mathbb{F} = \mathbb{C}$. Then $\mathcal{E} = \sum_i n_i\mathfrak{S}_i$ where all $\mathfrak{S}_i$ are simple A-modules. Then we have

Proposition 36. *$\Pi_{\mathfrak{S}_i} \mid_{\mathfrak{S}_j} = \delta_{ij}id$*

Proof. $\Pi_{\mathfrak{S}_i} \mid_{\mathfrak{S}_j} \in Hom_A(\mathfrak{S}_j, \mathfrak{S}_j)$. But this means that $\Pi_{\mathfrak{S}_i} \mid_{\mathfrak{S}_j} (y) \in Hom_{\mathbb{C}}(\mathfrak{S}_{jy}, \mathfrak{S}_{jy})$ for all $y \in \mathcal{S}$. From the Schur lemma we can conclude that $\Pi_{\mathfrak{S}_i} \mid_{\mathfrak{S}_j} (y) = \lambda_{ij}(y)id_y$. But then

$$dim\mathfrak{S}_{jy}\lambda_{ij}(y) = Tr(\Pi_{\mathfrak{S}_i} \mid_{\mathfrak{S}_j} (y)) = \frac{dim\mathfrak{S}_{iy}}{\mid H_y \mid} \sum_{h_y\in H_y} \chi_{iy}(h_y^{-1})\chi_{jy}(h_y) = \delta_{ij}dim\mathfrak{S}_{iy}$$

So we have $\lambda_{ij}(y) = \delta_{ij}$ and the proof is complete. □

Corollary 10. $\Pi_{\mathfrak{S}_i}$ *is a projection of* $\mathcal{E}$ *onto* $n_i\mathfrak{S}_i \subset \mathcal{E}$.

The projection formula can be used to simplify the solution process for GF-difference equations.

Proposition 37. ϕ *is a solution of* $\mathcal{E}$ *of type* $\mathfrak{S}$ *if and only of* $\phi = \Pi^*_{\mathfrak{S}}(\psi)$ *for some solution* ψ *of* $\Pi_{\mathfrak{S}}(\mathcal{E})$ *of type* $\mathfrak{S}$.

Proof. Let $\psi \in Hom_A(n_i\mathfrak{S}_i, \mathfrak{S}_i)$. Then $\Pi^*_{\mathfrak{S}_i}(\psi) = \psi \circ \Pi_{\mathfrak{S}_i} \in Hom_A(\mathcal{E}, \mathfrak{S}_i)$ and so $\Pi^*_{\mathfrak{S}_i}(\psi)$ is a solution of $\mathcal{E}$ of type $\mathfrak{S}_i$. Conversely let $\phi \in Hom_A(\mathcal{E}, \mathfrak{S}_i)$ be a solution of $\mathcal{E}$ of type $\mathfrak{S}_i$. We know by the structure theorem that $\mathcal{E} \approx \sum_j n_j\mathfrak{S}_j$. By the Schur lemma $\phi\mid_{n_j\mathfrak{S}_j} = 0$ for $i \neq j$. Let $\psi = \phi\mid_{n_i\mathfrak{S}_i}$, then $\psi \in Hom_A(n_i\mathfrak{S}_i, \mathfrak{S}_i)$ is a solution of $n_i\mathfrak{S}_i$ of type $\mathfrak{S}_i$ and for any $e = \sum_j e_j$ in $\mathcal{E} \approx \sum_j n_j\mathfrak{S}_j$ we have

$$\Pi^*_{\mathfrak{S}_i}(\psi)(e) = \psi(\Pi_{\mathfrak{S}_i}(e)) = \psi(e_i) = \phi(e_i) = \phi(e).$$

□

6. Coordinate description of GF-difference equations

Let $\mathcal{E}$ be any GF-difference equation. The structure of $\mathcal{E}$ is essentially determined by the action of G on the k-module $\mathcal{E}$. Let $\{e_i\}_{i=1}^n$ be a k-basis for $\mathcal{E}$. Define a set of matrices $\mathcal{E}^g \in Mat(n,k)$ by

$$ge_i = \sum_j \mathcal{E}^g_{ij} e_j$$

The set of matrices $\{\mathcal{E}^g\}_{g\in G}$ determines the G-action on $\mathcal{E}$ with respect to the gives k-basis. They formally play the same role as the connection symbols in differential geometry and we will call them *the connection of the given GF-difference equation.* In general $\mathcal{E}^{gg'} \neq \mathcal{E}^g\mathcal{E}^{g'}$ so the relation $g \longrightarrow \mathcal{E}^g$ is not a representation of G. We have however the following result.

Proposition 38. $\mathcal{E}^{gg'} = g(\mathcal{E}^{g'})\mathcal{E}^g$.

Proof.

$$\begin{aligned}(gg')e_i = g(g'e_i) = g\sum_j \mathcal{E}^{g'}_{ij} e_j = \sum_j g(\mathcal{E}^{g'}_{ij}) g e_j \\ = \sum_j \sum_k g(\mathcal{E}^{g'}_{ij})\mathcal{E}^g_{jk} e_k = \sum_j (\sum_k g(\mathcal{E}^{g'}_{ik})\mathcal{E}^g_{kj}) e_j.\end{aligned}$$

So we can conclude that

$$\mathcal{E}^{gg'}_{ij} = \sum_k g(\mathcal{E}^{g'}_{ik})\mathcal{E}^g_{kj}$$

□

From this result we immediately have

Corollary 11. $(\mathcal{E}^g)^{-1} = g(\mathcal{E}^{g^{-1}})$.

6.1. **Coordinate description of tensor operations.** We will now investigate how the connections of GF-difference equations behave when we perform the usual linear algebra operations on the corresponding modules.

Let $\mathcal{E}$ and $\mathcal{F}$ be left A-modules and let $\{e_i\}_{i=1}^{n}, \{f_j\}_{j=1}^{m}$ be k-basises for $\mathcal{E}$ and $\mathcal{F}$. Let $\{\mathcal{E}^g\}_{g\in G}$ and $\{\mathcal{F}^g\}_{g\in G}$ be the connection of the G-difference equations $\mathcal{E}$ and $\mathcal{F}$ with respect to the given basis. With respect to direct sum we have the following result

Proposition 39. $(\mathcal{E}\oplus\mathcal{F})^g = \mathcal{E}^g\oplus\mathcal{F}^g$.

Proof. A basis for $\mathcal{E}\oplus\mathcal{F}$ is $\{(e_i,0),(0,f_j)\}_{i=1,j=1}^{n,m}$ and

$$\begin{aligned} g(e_i,0) &= (ge_i,0) = (\sum_j \mathcal{E}^g_{ij}e_j,0) \\ &= \sum_j \mathcal{E}^g_{ij}(e_j,0), \\ g(0,f_i) &= (0,gf_i) = (0,\sum_j \mathcal{F}^g_{ij}f_j) \\ &= \sum_j \mathcal{F}^g_{ij}(0,f_j). \end{aligned}$$

So we have proved that $(\mathcal{E}\oplus\mathcal{F})^g = \mathcal{E}^g\oplus\mathcal{F}^g$. □

For k-tensorproduct we have a similar simple result.

Proposition 40. $(\mathcal{E}\otimes_k\mathcal{F})^g = \mathcal{E}^g\otimes_k\mathcal{F}^g$.

Proof. Since $\mathcal{E}$ and $\mathcal{F}$ are free k-modules it follows that $\{e_i\otimes_k f_j\}_{i=1,j=1}^{n,m}$ is a k-basis for $\mathcal{E}\otimes_k\mathcal{F}$. Furthermore we have

$$\begin{aligned} g(e_i\otimes_k f_j) &= ge_i\otimes_k gf_j = (\sum_r \mathcal{E}^g_{ir}e_r)\otimes_k(\sum_s \mathcal{F}^g_{js}f_s) \\ &= \sum_r\sum_s \mathcal{E}^g_{ir}\mathcal{F}^g_{js}e_r\otimes_k f_s. \end{aligned}$$

So

$$(\mathcal{E}\otimes_k\mathcal{F})^g_{irjs} = \mathcal{E}^g_{ir}\mathcal{F}^g_{js}.$$

□

For $Hom_k(\mathcal{E},\mathcal{F})$ we have the following result.

Proposition 41. $Hom_k(\mathcal{E},\mathcal{F})^g = \mathcal{F}^g\otimes_k((\mathcal{E}^g)^t)^{-1}$.

Proof. Define elements $\delta_{ij}\in Hom(\mathcal{E},\mathcal{F})$ by

$$\delta_{ij}(e_k) = \begin{cases} f_i & \text{if } j=k, \\ 0 & \text{if } j\neq k. \end{cases}$$

Then $\{\delta_{ij}\}$ is a k-basis for $Hom_k(\mathcal{E},\mathcal{F})$. Furthermore we have

$$\begin{aligned}(g\delta_{ij})(e_u) &= g(\delta_{ij}(g^{-1}e_u)) = g(\delta_{ij}(\sum_v \mathcal{E}^{g^{-1}}_{uv} e_v)) \\ &= g\sum_v \mathcal{E}^{g^{-1}}_{uv}\delta_{ij}(e_v) = g(\mathcal{E}^{g^{-1}}_{uj})gf_i \\ &= \sum_r g(\mathcal{E}^{g^{-1}}_{uj})\mathcal{F}^g_{ir}f_r.\end{aligned}$$

So we can conclude that

$$Hom_k(\mathcal{E},\mathcal{F})^g = \mathcal{F}^g \otimes_k (g(\mathcal{E}^{g^{-1}}))^t.$$

But $g\mathcal{E}^{g^{-1}} = (\mathcal{E}^g)^{-1}$ and the proof is complete. □

Consider the special case $\mathcal{E}^* = Hom_k(\mathcal{E},k)$. Clearly $\mathcal{F}^g = k^g = 1$ and the previous proposition gives

Corollary 12. *$(\mathcal{E}^*)^g = ((\mathcal{E}^g)^t)^{-1}$.*

For symmetric and antisymmetric product there are no simple general formulas for computing the connection of $S^n\mathcal{E}$ and $\wedge^n\mathcal{E}$ in terms of the connection of $\mathcal{E}$. For notational simplicity we only consider the case $n = 2$. Let $\{e_i\}$ be a k-basis for $\mathcal{E}$. Then $\{e_ie_j\}_{i\leqslant j}$ is a k-basis for $S^2\mathcal{E}$. We have by definition

$$ge_i = \sum_j \mathcal{E}^g_{ij}e.j$$

But then we have

$$\begin{aligned}g(e_ie_j) = ge_ige_j &= (\sum_k \mathcal{E}^g_{ik}e_k)(\sum_l \mathcal{E}^g_{jl}e_l) = \sum_{kl}\mathcal{E}^g_{ik}\mathcal{E}^g_{jl}e_ke_l \\ &= \sum_{k>l}\mathcal{E}^g_{ik}\mathcal{E}^g_{jl}e_ke_l + \sum_{k=l}\mathcal{E}^g_{ik}\mathcal{E}^g_{jl}e_ke_l + \sum_{k<l}\mathcal{E}^g_{ik}\mathcal{E}^g_{jl}e_ke_l \\ &= \sum_k \mathcal{E}^g_{ik}\mathcal{E}^g_{jk}e_ke_k + \sum_{k<l}(\mathcal{E}^g_{ik}\mathcal{E}^g_{jl} + \mathcal{E}^g_{il}\mathcal{E}^g_{jk})e_ke_l\end{aligned}$$

So for $n = 2$ we have

$$(S^2\mathcal{E})^g_{ijkl} = \begin{cases}\mathcal{E}^g_{ik}\mathcal{E}^g_{jk} & k = l \\ \mathcal{E}^g_{ik}\mathcal{E}^g_{jl} + \mathcal{E}^g_{il}\mathcal{E}^g_{jk} & k < l.\end{cases}$$

We now consider $\wedge^2\mathcal{E}$. A k-basis for $\wedge^2\mathcal{E}$ is $\{e_i\wedge e_j\}_{i<j}$. Furthermore we have

$$\begin{aligned}g(e_i\wedge e_j) = ge_i\wedge ge_j &= (\sum_k \mathcal{E}^g_{ik}e_k)\wedge(\sum_l \mathcal{E}^g_{jl}e_l) = \sum_{kl}\mathcal{E}^g_{ik}\mathcal{E}^g_{jl}e_k\wedge e_l \\ &= \sum_{k<l}\mathcal{E}^g_{ik}\mathcal{E}^g_{jl}e_k\wedge e_l + \sum_{k>l}\mathcal{E}^g_{ik}\mathcal{E}^g_{jl}e_k\wedge e_l \\ &= \sum_{k<l}(\mathcal{E}^g_{ik}\mathcal{E}^g_{jl} - \mathcal{E}^g_{il}\mathcal{E}^g_{jk})e_k\wedge e_l.\end{aligned}$$

We can conclude that

$$(\wedge^2\mathcal{E})^g_{ijkl} = \mathcal{E}^g_{ik}\mathcal{E}^g_{jl} - \mathcal{E}^g_{il}\mathcal{E}^g_{jk}.$$

Assume that $dim_k\mathcal{E} = 2$. Then we have

$$\mathcal{E}^g = \begin{pmatrix} \mathcal{E}^g_{11} & \mathcal{E}^g_{12} \\ \mathcal{E}^g_{21} & \mathcal{E}^g_{22} \end{pmatrix}$$

The module $\wedge^2\mathcal{E}$ is one dimensional over k with basis $e_1 \wedge e_2$. The matrix $(\wedge^2\mathcal{E})^g$ is the scalar

$$(\wedge^2\mathcal{E})^g = \mathcal{E}^g_{11}\mathcal{E}^g_{22} - \mathcal{E}^g_{12}\mathcal{E}^g_{21} = det(\mathcal{E}^g).$$

It is evident that a similar result holds in general

Proposition 42. *Let $dim_k\mathcal{E} = n$. Then*

$$(\wedge^n\mathcal{E})^g = det(\mathcal{E}^g).$$

6.2. **Coordinate description of A-module morphisms and solutions.** Let $\mathcal{E}$ and $\mathcal{F}$ be GF-difference equations with k-bases $\{e_i\}_{i=1}^n, \{f_i\}_{i=1}^m$ and connection $\{\mathcal{E}^g\}_{g\in G}, \{\mathcal{F}^g\}_{g\in G}$. Let $\phi \in Hom_A(\mathcal{E},\mathcal{F})$ be a A-module morphism. Then ϕ is k-linear and has a matrix $\phi = (\phi_{ij})$ with respect to the given bases for $\mathcal{E}$ and $\mathcal{F}$.

Proposition 43. *$g(\phi)\mathcal{F}^g = \mathcal{E}^g\phi$ for all $g \in G$.*

Proof. The k-morphism ϕ is a A-module morphism only if $\phi(ge) = g\phi(e)$ for all $g \in G$. But we have

$$\begin{aligned} \phi(ge_i) &= \phi(\sum_j \mathcal{E}^g_{ij}e_j) = \sum_j \mathcal{E}^g_{ij}\phi(e_j) = \sum_j \mathcal{E}^g_{ij}\sum_k \phi_{jk}f_k \\ &= \sum_k(\sum_j \mathcal{E}^g_{ij}\phi_{jk})f_k, \\ g\phi(e_i) &= g(\sum \phi_{ij}f_j) = \sum_j g(\phi_{ij})gf_j = \sum_j g(\phi_{ij})\sum_k \mathcal{F}^g_{jk}f_k \\ &= \sum_k(\sum_j g(\phi_{ij})\mathcal{F}^g_{jk})f_k \end{aligned}$$

comparing sides we have $\sum_j g(\phi_{ij})\mathcal{F}^g_{jk} = \sum_j \mathcal{E}^g_{ij}\phi_{jk}$ for all i,k and $g \in G$. □

It is clearly sufficient that the matrix ϕ satisfie the equation in the previous proposition only on a set of generators for G. We have defined solution of type $\mathcal{F}$ to a given GF-difference equation $\mathcal{E}$ as A-morphisms from $\mathcal{E}$ to $\mathcal{F}$. Using coordinates as in the previous proposition we see that solutions in our sense is a matrix of functions on $\mathcal{S}$ that solves a set of classical difference equations. This show that we are not redefining the notion of solution and justifies our use of this term in our theory.

7. Invariant structures

Let $\mathcal{E},\mathcal{E}',\mathcal{F}$ and $\mathcal{F}'$ be GF-difference equations. Let $\phi \in Hom_A(\mathcal{E},\mathcal{F})$ and $\psi \in Hom_A(\mathcal{E}',\mathcal{F}')$ and define maps $\phi\otimes_k\psi, S^n\phi, \wedge^n\phi$ by

$$\begin{aligned} (\phi\otimes_k\psi)(e\otimes_k e') &= \phi(e)\otimes_k\psi(e'), \\ S^n\phi(e_1e_2\cdots e_n) &= \phi(e_1)\phi(e_2)\cdots\phi(e_n), \\ \wedge^n\phi(e_1\wedge e_2\cdots\wedge e_n) &= \phi(e_1)\wedge\phi(e_2)\cdots\wedge(e_n). \end{aligned}$$

Then the maps are well defined and we have the following result whose proof can be found in standard texts [?].

Proposition 44. *The maps $\phi \otimes_k \psi, S^n\phi, \wedge^n\phi$ are well defined and*

$$\phi \otimes_k \psi \in Hom_A(\mathcal{E} \otimes_k \mathcal{E}', \mathcal{F} \otimes_k \mathcal{F}'),$$
$$S^n\phi \in Hom_A(S^n\mathcal{E}, S^n\mathcal{F}),$$
$$\wedge^n\phi \in Hom_A(\wedge^n\mathcal{E}, \wedge^n\mathcal{F}).$$

7.1. **Conserved quantities.** Let $\mathcal{E}, \mathcal{F}$ be GF-difference equations and let $\phi \in Hom_A(\mathcal{E}, \mathcal{F})$ be a solution of $\mathcal{E}$ of type $\mathcal{F}$. Using ϕ we can generate morphisms $S^n\phi : S^n\mathcal{E} \longrightarrow S^n\mathcal{F}$ and $\wedge^n\phi : \wedge^n\phi \longrightarrow \wedge^n\mathcal{F}$. Let us first consider the symmetric case. Assume that there exists an invariant element or structure $\alpha \in S^n\mathcal{E}$. This means that $g\alpha = \alpha$ for all $g \in G$. Then $S^n\phi(\alpha) \in S^n\mathcal{F}$ and we have

$$g(S^n\phi(\alpha)) = S^n\phi(g\alpha) = S^n\phi(\alpha)$$

So that $S^n\phi(\alpha)$ is an invariant structure in $S^n\mathcal{F}$. Let us consider the particular case when $\mathcal{F}$ is the simple object $\mathfrak{S} \approx k$ corresponding to the trivial action of G. Then $S^n\mathfrak{S} \approx k$ and we have the following result

Proposition 45. *Let $\mathcal{E}$ be any GF-difference equation with an invariant structure $\alpha \in S^n\mathcal{E}$. Let $\phi \in Hom_A(\mathcal{E}, k)$ be any solution of $\mathcal{E}$ of type $\mathfrak{S} \approx k$. Then*

$$S^n\phi(\alpha) = constant.$$

Proof. We know that $S^n\phi(\alpha)$ is an element in k and that $gS^n\phi(\alpha) = S^n\phi(\alpha)$ for all elements in G. But G acts transitivly so $S^n\phi(\alpha)$ must be a constant function. □

Note that in coordinates this gives us a symmetric polynomial invariant for the equation $\mathcal{E}$.

Let us next consider the antisymmetric case. Let $\alpha \in \wedge^n\mathcal{E}$ be a invariant structure for $\mathcal{E}$ so that $g\alpha = \alpha$ for all $g \in G$. Then $g(\wedge^n\phi(\alpha)) = \wedge^n\phi(\alpha)$ so $\wedge^n\phi(\alpha)$ is an invariant structure in $\wedge^n\mathcal{F}$. This leads to the following proposition

Proposition 46. *Let $\mathfrak{S}$ be a GF-difference equation and $n = dim_k\mathcal{F}$. Let $\mathcal{E}$ be any GF-difference equation with invariant structure $\alpha \in \wedge^n\mathcal{E}$. Then*

$$\wedge^n\phi(\alpha) = constant.$$

for any solution $\phi \in Hom_A(\mathcal{E}, \mathfrak{S})$.

This gives us an antisymmetric polynomial invariant for the equation $\mathcal{E}$. In a similar way conditions for other types of conservation laws can be specified through invariants.

7.2. **Self-dual equations.** Let $\mathcal{E}$ be any GF-difference equation and let $\mathcal{E}^*$ be the dual equation. Assume that there is an invariant structure $\alpha \in S^2\mathcal{E}^*$ or $\alpha \in \wedge^2\mathcal{E}^*$. Define a map $F_\alpha : \mathcal{E} \longrightarrow \mathcal{E}'$ by

$$F_\alpha(e)(e') = \alpha(e, e').$$

We have the following result

Proposition 47. *F_α is a A-morphism.*

Proof. It is evident that $F_\alpha(e) \in \mathcal{E}^*$ and that F_α is k-linear. Furthermore we have

$$\begin{aligned} F_\alpha(ge)(e') &= \alpha(ge, e') = gg^{-1}(\alpha(ge, g(g^{-1}e'))) = g((g^{-1}\alpha)(e, g^{-1}e')) \\ &= g(\alpha(e, g^{-1}e')) = g(F_\alpha(e)(g^{-1}e')) = (gF_\alpha(e))(e'). \end{aligned}$$

□

Let us now define the notion of self duality for GF-difference equations.

Definition 8. A GF-difference equation $\mathcal{E}$ is self-dual if $\mathcal{E} \approx \mathcal{E}^*$ as A-modules.

Using the previous proposition we can now prove the following

Proposition 48. *Let $\mathcal{E}$ be a GF-difference equation and assume $\mathcal{E}$ has a nondegenerate invariant structure $\alpha \in S^2\mathcal{E}^*$ or $\alpha \in \wedge^2\mathcal{E}^*$. Then $\mathcal{E}$ is self-dual.*

Proof. We have a A-morphism $F_\alpha : \mathcal{E} \longrightarrow \mathcal{E}^*$. This map is bijective if α is nondegenerate because $F_\alpha(e) = \beta$ if and only if $\alpha(e, e') = \beta(e')$ for all e' and these equations has one and only one solution e since α is nondegenerate. □

This proposition show that any GF-difference equation with an invariant euclidian or symplectic structure is self-dual.

7.3. **Solutions and composition principles.** Let $\mathcal{E}, \mathcal{F}$ be GF-difference equations and let $\phi \in Hom_k(\mathcal{E}, \mathcal{F})$. Then ϕ is a solution of type $\mathcal{F}$ of $\mathcal{E}$ if $g\phi = \phi$ for all $g \in G$. This means that a solution is a invariant structure in $Hom_k(\mathcal{E}, \mathcal{F})$. Let $\alpha \in S^2(Hom_k(\mathcal{E}, \mathcal{F})^*) \otimes_k Hom_k(\mathcal{E}, \mathcal{F})$ be a invariant structure. Using the standard isomorphism $Hom_k(\mathcal{F}, \mathcal{F}') \approx \mathcal{F}^* \otimes_k \mathcal{F}'$, α defines a map $T_\alpha : Hom_k(\mathcal{E}, \mathcal{F}) \otimes_k Hom_k(\mathcal{E}, \mathcal{F}) \longrightarrow Hom_k(\mathcal{E}, \mathcal{F})$ defined by

$$T_\alpha(\phi, \psi) = \alpha(\phi, \psi)$$

For the map T_α we have the following result

Proposition 49. *Let $\mathcal{E}, \mathcal{F}$ be a GF-difference equations and. Let $\phi, \psi \in Hom_A(\mathcal{E}, \mathcal{F})$ be a pair of solutions of $\mathcal{E}$ of type $\mathcal{F}$. Then $T_\alpha(\phi, \psi) \in Hom_A(\mathcal{E}, \mathcal{F})$ is a solution of $\mathcal{E}$ of type $\mathcal{F}$.*

Proof. We have $g\phi = \phi$ and $g\psi = \psi$ for all $g \in G$ since they are solutions. But then we have

$$\begin{aligned} g(T_\alpha(\phi, \psi)) &= g(\alpha(g^{-1}g\phi, g^{-1}g\psi)) = (g\alpha)(g\phi, g\psi) \\ &= \alpha(\phi, \psi) = T_\alpha(\phi, \psi). \end{aligned}$$

□

So a GF-difference equation $\mathcal{E}$ has a symmetric composition principle for solutions of type $\mathcal{F}$ if there is an invariant structure in $S^2(Hom_k(\mathcal{E}, \mathcal{F})^*) \otimes_k Hom_k(\mathcal{E}, \mathcal{F})$. In a similar way other types of composition principles will correspond to the existense of certain invariants in the tensor algebra of the equation $\mathcal{E}$.

8. Module description of classical difference equations

We will now develope the analog of differential operators on sections in vectorbundles. Many of the constructions introduced also applies in the case of equations that are not of finite type. We will however in this section assume that all modules that appears are GF-difference equations. This will in particular mean that A itself must be a GF-difference equation. This can only happend if G is a finite group. Since G acts transitively on $\mathcal{S}$ this means that we are considering the situation where $\mathcal{S}$ is a finite set.

8.1. **The module of difference operators.** Let $\mathcal{E}$,$\mathcal{E}'$ be A-modules. We will define an action of elements in the module $Hom_k(\mathcal{E},\mathcal{E}') \otimes_k A$ on $\mathcal{E}$. For each element $(\phi, a) \in Hom_k(\mathcal{E},\mathcal{E}') \times A$ define a map $\mu(\phi, a) : \mathcal{E} \longrightarrow \mathcal{E}'$ by

$$\mu(\phi, a)(e) = \phi(ae).$$

Proposition 50. *$\mu(\phi, a)$ is $\mathbb{F}$-linear.*

Proof.

$$\begin{aligned}
\mu(\phi, a)(e + e') &= \phi(a(e + e')) = \phi(ae + ae') = \phi(ae) + \phi(ae') \\
&= \mu(\phi, a)(e) + \mu(\phi, a)(e'), \\
\mu(\phi, a)(re) &= \phi(a(re)) = \phi(r(ae)) = r\phi(ae) = r\mu(\phi, a)(e).
\end{aligned}$$

□

Let μ be the map $(\phi, a) \longrightarrow \mu(\phi, a)$. Then we have

Proposition 51. *μ is k-bilinear.*

Proof.

$$\begin{aligned}
\mu(\phi + \phi', a)(e) &= (\phi + \phi')(ae) = \phi(ae) + \phi'(ae) \\
&= \mu(\phi, a)(e) + \mu(\phi', a)(e), \\
\mu(\phi, a + a')(e) &= \phi((a + a')e) = \phi(ae + a'e) = \phi(ae) + \phi(a'e) \\
&= \mu(\phi, a)(e) + \mu(\phi, a')(e), \\
\mu(f\phi, a)(e) &= (f\phi)(ae) = f(\phi(ae)) = \phi(f(ae)) \\
&= \phi((fa)e) = \mu(\phi, fa)(e).
\end{aligned}$$

□

So we have a well defined map $\mu : Hom_k(\mathcal{E},\mathcal{E}') \otimes_k A \longrightarrow Hom_{\mathbb{F}}(\mathcal{E},\mathcal{E}')$ defined by

$$\mu(\phi \otimes_k a) = \phi(ae)$$

Proposition 52. *The map μ is a A-module morphism.*

Proof. By construction the map μ is a k-module morphism. Let $g \in G$, then we have

$$\begin{aligned}
\mu(g(\phi \otimes_k a))(e) &= \mu(g\phi \otimes_k ga) = (g\phi)(gae) \\
&= g(\phi(g^{-1}(gae))) = g(\phi(ae)) = g(\mu(\phi \otimes_k a)(e)) \\
&= (g\mu(\phi \otimes_k a))(e).
\end{aligned}$$

□

The elements $\theta \in Hom_k(\mathcal{E},\mathcal{E}') \otimes_k A$ thus acts as $\mathbb{F}$-linear maps from the module $\mathcal{E}$ to the module $\mathcal{E}'$. The action is defined by

$$\theta(e) = \mu(\theta)(e).$$

We will now consider the coordinate expression for these maps. Let $\{e_i\}, \{e'_i\}$ be k-basises for $\mathcal{E}$ and $\mathcal{E}'$. Then $\{\phi_{ij}\}$ is a basis for $Hom_k(\mathcal{E},\mathcal{E}')$ where $\phi_{ij}(e_k) = \delta_{ik}e'_j$.

Let $\theta \in Hom_k(\mathcal{E},\mathcal{E}') \otimes_k A$ and $e \in \mathcal{E}$. Using basis we have $\theta = \sum_{ijg} \theta_{ijg}\phi_{ij} \otimes_k g$ and $e = \sum_i f_i e_i$. This gives us

$$\begin{aligned}(\sum_{ijg} \theta_{ijg}\phi_{ij} \otimes_k g)(\sum_k e_k) &= \sum_{ijg} \theta_{ijg}\phi_{ij}(g \sum_k f_k e_k) = \sum_{ijg} \theta_{ijg}\phi_{ij}(\sum_k g(f)_k \sum_l \mathcal{E}^g_{kl} e_l) \\ &= \sum_{ijklg} \theta_{ijg} g(f)_k \mathcal{E}^g_{kl}\phi_{ij}(e_l) = \sum_{ijklg} \theta_{ijg} g(f)_k \mathcal{E}^g_{kl}\delta_{il} e'_j \\ &= \sum_{ijkg} \theta_{ijg} g(f)_k \mathcal{E}^g_{ki} e'_j.\end{aligned}$$

The equation $\theta(e) = 0$ is therefore equivalent to a system of classical difference equations

$$\sum_k (\sum_{ig} \theta_{ijg}\mathcal{E}^g_{ki} g) f_k = 0$$

In general any elements in the kernel of μ will be trivial when considered as $\mathbb{F}$-linear maps.

Example. Let $\mathcal{S} = \{x, y, z\}$ be the cyclic graph of three elements with symmetry group S_3. Let the group elements in cycle notation be $g_0 = id, g_1 = (1,3,2), g_2 = (1,2,3), g_3 = (1,2), g_4 = (2,3)$ and $g_5 = (1,3)$. Then the element $\theta = \phi \otimes_k (g_0 + g_1 + g_2 - g_3 - g_4 - g_5)$, $\phi \neq 0$, is trivial as a $\mathbb{F}$-linear map.

We therefore makes the following definition

Definition 9. $Difn_*(\mathcal{E},\mathcal{E}') = (Hom_k(\mathcal{E},\mathcal{E}') \otimes_k A)/ker\mu$ is the module of difference operators from $\mathcal{E}$ to $\mathcal{E}'$.

8.2. **Composition of difference operators.** Let $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_3$ be A-modules. For each pair of elements $(\phi, g) \in Hom_k(\mathcal{E}_2,\mathcal{E}_3) \times A$ define a map $F^\phi_g : Hom_k(\mathcal{E}_1,\mathcal{E}_2) \times A \longrightarrow Hom_k(\mathcal{E}_1,\mathcal{E}_3) \otimes_k A$ by

$$F^\phi_g(\psi, b) = \phi \circ g\psi \otimes_k gb$$

Proposition 53. *F^ϕ_g is middle k-linear for each $(\phi, g) \in Hom_k(\mathcal{E}_2,\mathcal{E}_3) \times A$.*

Proof.

$$\begin{aligned}F^\phi_g(\psi + \psi', b) &= \phi \circ (g(\psi + \psi')) \otimes_k gb = \phi \circ (g\psi + g\psi') \otimes_k gb = (\phi \circ g\psi + \phi \circ g\psi') \otimes_k gb \\ &= \phi \circ g\psi \otimes_k gb + \phi \circ g\psi' \otimes_k gb = F^\phi_g(\psi, b) + F^\phi_g(\psi', b), \\ F^\phi_g(\psi, b + b') &= \phi \circ g\psi \otimes_k g(b + b') = \phi \circ g\psi \otimes_k (gb + gb') \\ &= \phi \circ g\psi \otimes_k gb + \phi \circ g\psi \otimes_k gb' = F^\phi_g(\psi, b) + F^\phi_g(\psi, b'), \\ F^\phi_g(\psi, fb) &= \phi \circ g\psi \otimes_k g(fb) = \phi \circ g\psi \otimes_k g(f)gb = g(f)(\phi \circ g\psi) \otimes_k gb \\ &= \phi \circ (g(f)(g\phi)) \otimes_k gb = \phi \circ ((g(f)g)\psi) \otimes_k gb = \phi \circ ((g(f))\psi) \otimes_k gb \\ &= \phi \circ g(f\psi) \otimes_k gb = F^\phi_g(f\psi, b).\end{aligned}$$

∎

So we have a well defined $\mathbb{F}$-linear map $F_g^{\phi} : Hom_k(\mathcal{E}_1, \mathcal{E}_2) \otimes_k A \longrightarrow Hom_k(\mathcal{E}_1, \mathcal{E}_3) \otimes_k A$ defined by

$$F_g^{\phi}(\psi \otimes_k b) = \phi \circ g\psi \otimes_k gb.$$

We use this map to define a map $F : Hom_k(\mathcal{E}_2, \mathcal{E}_3) \times A \longrightarrow Hom_{\mathbb{F}}(Hom_k(\mathcal{E}_1, \mathcal{E}_2) \otimes_k A, Hom_k(\mathcal{E}_1, \mathcal{E}_3) \otimes_k A)$ by

$$F(\phi, a) = \sum_g a_g F_g^{\phi}$$

where $a = \sum_g a_g g$.

Proposition 54. *F is k-bilinear.*

Proof. We have

$$\begin{aligned} F_g^{\phi+\phi'}(\psi \otimes_k b) &= (\phi + \phi') \circ g\psi \otimes_k gb = (\phi \circ g\psi + \phi' \circ g\psi) \otimes_k gb \\ &= \phi \circ g\psi \otimes_k gb + \phi' \circ g\psi \otimes_k gb = F_g^{\phi}(\psi \otimes_k b) + F_g^{\phi'}(\psi \otimes_k b). \end{aligned}$$

Using this we find

$$\begin{aligned} F(\phi + \phi', a) &= \sum_g a_g F_g^{\phi+\phi'} = \sum_g a_g F_g^{\phi} + \sum_g a_g F_g^{\phi'} = F(\phi, a) + F(\phi', a), \\ F(\phi, a + a') &= \sum_g (a_g + a'_g) F_g^{\phi} = \sum_g a_g F_g^{\phi} + \sum_g a'_g F_g^{\phi} = F(\phi, a) + F(\phi, a'), \\ F(f\phi, a)(\psi \otimes_k b) &= \sum_g a_g F_g^{f\phi}(\psi \otimes_k b) = \sum_g a_g((f\phi) \circ g\psi \otimes_k gb) \\ &= \sum_g a_g(f(\phi \circ g\psi) \otimes_k gb) = \sum_g a_g f(\phi \circ g\psi \otimes_k gb) \\ &= \sum_g (fa_g)(\phi \circ g\psi \otimes_k gb). = F(\phi, fa)(\psi \otimes_k b) \end{aligned}$$

□

We can conclude that we have a well defined map $F : Hom_k(\mathcal{E}_2, \mathcal{E}_3) \otimes_k A \longrightarrow Hom_{\mathbb{F}}(Hom_k(\mathcal{E}_1, \mathcal{E}_2) \otimes_k A, Hom_k(\mathcal{E}_1, \mathcal{E}_3) \otimes_k A)$

$$F(\phi \otimes_k a)(\psi \otimes_k b) = \sum_g a_g \phi \circ (g\psi) \otimes_k gb$$

Let $\mu_i, i = 1, 2, 3$ be the action map. Then we have

Proposition 55. *$F(\theta_2)(\theta_1) \in ker\mu_3$ if $\theta_1 \in ker\mu_1$ or $\theta_2 \in ker\mu_2$*

Proof. Let $\theta_1 \in Hom_k(\mathcal{E}_1, \mathcal{E}_2) \otimes_k A, \theta_2 \in Hom_k(\mathcal{E}_2, \mathcal{E}_3) \otimes_k A$. Then $\theta_1 = \sum_i \psi_i \otimes_k b_i$ and $\theta_2 = \sum_i \phi_i \otimes_k a_i$. But then we have

$$F(\theta_2)(\theta_1) = \sum_{ij} F(\phi_i \otimes_k a_i)(\psi_j \otimes_k b_j) = \sum_{ijg} a_{ig} \phi_i \circ g\psi_j \otimes_k gb_j$$

Using this we find

$$\begin{aligned}
\mu_3(F(\theta_2)(\theta_1))(e_1) &= \sum_{ijg} a_{ig}(\phi \circ g\psi_j)(gb_j e_1) = \sum_{ijg} a_{ig}\phi_i(g\psi_j(b_j e_1)) \\
&= \sum_{ig} a_{ig}\phi_i(g(\sum_j \psi_j(b_j e_1))) = \sum_{ig} a_{ig}\phi_i(g\mu_1(\theta_1)(e_1) \\
&= \sum_i (\phi_i(\sum_g a_{ig}\mu_1(\theta_1)(e_1))) = \sum_i \phi_i(a_i\mu_1(\theta_1)(e_1)) \\
&= \mu_2(\theta_2)(\mu_1(\theta_1)(e_1)) = 0
\end{aligned}$$

if $\theta_1 \in ker\mu_1$ or $\theta_2 \in ker\mu_2$ □

The map F therefore restricts to the modules of difference operators and we have a map $c : Difn_*(\mathcal{E}_2, \mathcal{E}_3) \times Difn_*(\mathcal{E}_1, \mathcal{E}_2) \longrightarrow Difn_*(\mathcal{E}_1, \mathcal{E}_3)$ defined by

$$c([\theta_2], [\theta_1]) = [F(\theta_2)(\theta_1)].$$

We use the map c to define composition of difference operators.

Definition 10. Let $\Delta_1 \in Difn_*(\mathcal{E}_1, \mathcal{E}_2)$ and $\Delta_2 \in Difn_*(\mathcal{E}_2, \mathcal{E}_3)$ be difference operators. Define the composition $\Delta_2 \circ \Delta_1 \in Difn_*(\mathcal{E}_1, \mathcal{E}_3)$ by

$$\Delta_2 \circ \Delta_1 = c(\Delta_2, \Delta_1)$$

8.3. **Modules corresponding to difference operators.** By construction $Difn_*(k, k)$ is a left A-module. Let $\mu : A \longrightarrow Hom_{\mathbb{F}}(k, k)$ be the action map of A. Then by definition $Difn_*(k, k) = A/ker\mu$. For any element $a \in A$ let $\Delta_a = [id \otimes_k a] \in Difn_*(k, k)$ be the corresponding difference operator. Then we have

Proposition 56. *Let $\Delta \in Difn_*(\mathcal{E}, k)$. Then*

$$a\Delta = \Delta_a \circ \Delta.$$

Proof. We only need to consider a generating set for $Difn_*(\mathcal{E}, k)$. Let $\Delta = [\alpha \otimes_k b]$. Then we have

$$\begin{aligned}
a\Delta &= [a(\alpha \otimes_k b)] = [\sum_g a_g g(\alpha \otimes_k b)] = [\sum_g a_g(g\alpha \otimes_k gb)] \\
&= [\sum_g a_g F_g^{id}(\alpha \otimes_k b)] = [F(id \otimes_k a)(\alpha \otimes_k b)] = c([id \otimes_k a], [\alpha \otimes_k b]) \\
&= \Delta_a \circ \Delta.
\end{aligned}$$

□

Let $\Delta \in Difn_*(\mathcal{E}_1, \mathcal{E}_2)$. Define a map $\phi^\Delta : Difn_*(\mathcal{E}_2, k) \longrightarrow Difn_*(\mathcal{E}_1, k)$ by

$$\phi^\Delta(\nabla) = \nabla \circ \Delta$$

Then we have

Proposition 57. *ϕ^Δ is a left A-module morphism.*

Proof.

$$\phi^\Delta(a\nabla) = (a\nabla) \circ \Delta = (\Delta_a \circ \nabla) \circ \Delta = \Delta_a \circ (\nabla \circ \Delta) = a\phi^\Delta(\nabla)$$

□

Definition 11. Let $\Delta \in Difn_*(\mathcal{E}_1, \mathcal{E}_2)$. The GF-difference equation corresponding to Δ is

$$\mathcal{E}_\Delta = Coker \phi^\Delta.$$

8.4. **Classical solutions.** Let $\Delta \in Difn_*(\mathcal{E}_1, \mathcal{E}_2)$ be a difference operator. Define the set of classical solutions $C(\Delta)$ of Δ by

Definition 12. $C(\Delta) = \{e \in \mathcal{E}_1 \mid \Delta(e) = 0\}$.

Let $\mathfrak{S}_0 \approx k$ be the simple module corresponding to trivial action of G. For each $e \in C(\Delta)$ define a map $\phi_e : \mathcal{E}_\Delta \longrightarrow \mathfrak{S}_0$ by

$$\phi_e([\lambda]) = \lambda(e).$$

Proposition 58. *ϕ_e is well defined for each $e \in C(\Delta)$.*

Proof. Assume that $[\lambda] = [\lambda']$. Then $\lambda - \lambda' = \phi^\Delta(\nabla)$ for some $\nabla \in Difn_*(\mathcal{E}_2, k)$. But then we have

$$\begin{aligned}\phi_e([\lambda]) &= \lambda(e) = \lambda'(e) + \phi^\Delta(\nabla)(e) \\ &= \lambda'(e) + \nabla(\phi_\Delta(e)) = \lambda'(e).\end{aligned}$$

□

Proposition 59. $\phi_e \in Hom_A(\mathcal{E}_\Delta, \mathfrak{S}_0)$

Proof.

$$\begin{aligned}\phi_e(a[\lambda]) &= \phi_e([a\lambda]) = (\Delta_a \circ \lambda)(e) = \Delta_a(\lambda(e)) \\ &= (id \otimes_k a)(\lambda(e)) = a\lambda(e) = a\phi_e([\lambda]).\end{aligned}$$

□

Now define a map $\phi : C(\Delta) \longrightarrow Hom_A(\mathcal{E}_\Delta, \mathfrak{S}_0)$ by $\phi(e) = \phi_e$. Then we have

Proposition 60. *$\phi : C(\Delta) \longrightarrow \phi(C(\Delta)) \subset Hom_A(\mathcal{E}_\Delta, \mathfrak{S}_0)$ is a isomorphism of $\mathbb{F}$-vectorspaces.*

Proof. Assume that $\phi(e) = \phi(e')$. Then we have that $\phi_e([\lambda]) = \phi_{e'}([\lambda])$ so that $\lambda(e - e') = 0$ for all $\lambda \in Difn_*(\mathcal{E}_1, k)$. Let $\{e_i\}$ be a basis for $\mathcal{E}_1$ and $\{e_i^*\}$ the dual basis. Then $e - e' = \sum_i f_i e_i$, $\lambda_j = [e_j^* \otimes_k 1] \in Difn_*(\mathcal{E}_1, k)$ and we have $f_j = e_j^*(\sum_i f_i e_i) = \lambda_j(e - e') = 0$. So $e = e'$ and ϕ is injective. Furthermore we have $\phi(re)([\lambda]) = \lambda(re) = r\lambda(e) = r\phi(e)([\lambda])$ so ϕ is $\mathbb{F}$-linear. □

The previous proposition show that any classical solution of a difference operator Δ is contained in the set of solutions of $\mathcal{E}_\Delta$ of type $\mathfrak{S}_0$.

8.5. **Modules corresponding to systems of difference equations.** Any system of difference equations on the space $\mathcal{S}$ is of the form

$$\sum_{k=1}^{n} (\sum_g c_{kg}^j g) f_k = 0 \quad \text{for } j = 1 \cdots m$$

The given system of difference equations will only fix the k-module structure of the A-modules $\mathcal{E}_1$ and $\mathcal{E}_2$. It will not fix the A-module structure or the operator $\Delta \in Difn_*(\mathcal{E}_1, \mathcal{E}_2)$ separately but will fix a relation between the A-module structure on $\mathcal{E}_1$ and the operator Δ. The space of solutions of the given system of difference equations must be equal to $C(\Delta)$ Using bases $\{e_i\}$ and $\{f_i\}$ for $\mathcal{E}_1$ and $\mathcal{E}_2$ we have $\Delta = [\sum_{ijg} \theta_{ijg}\phi_{ij} \otimes_k g]$ and the relation is

$$\sum_i \theta_{ijg}\mathcal{E}^g_{ki} = c^j_{kg}$$

where $\mathcal{E}^g$ is the connection for the action of g on $\mathcal{E}_1$. This means that in general we have many different modules $\mathcal{E}_\Delta$ corresponding to a given system of difference equations. However for all these modules $\mathcal{E}_\Delta$ we have $C(\Delta) \in Hom_A(\mathcal{E}_\Delta, \mathfrak{S}_0)$ so they all contain the set of solutions of the given system of difference equations.

References

[1] "Differential algebra",J. F. Ritt,AMS Colloquim XXXIII,1950.
[2] "Differential algebra and algebraic groups",E. R. Kolchin,Academic Press, 1973.
[3] "Geometry of jet spaces and nonlinear partial differential equations",I. S. Krasil'shchik, V. V. Lychagin,A. M. Vinogradov, Gordon and Breach, 1986.
[4] " Difference algebra",R. M. Cohn,Interscience Publishers,1965.
[5] "Categories for the working mathematician,",S. Mac Lane,GTM 5,1971. bibitemrow "Ring theory",L. Rowen, Academic Press,1988.
[6] "Algebra",S. Lang,Addison Wesley,1993.
[7] "Character theory of finite groups",I. M. Isaacs,Academic Press,1976.
[8] "A course in the theory of groups",D. J. S. Robinson,GTM 80,1982.

Faculty of science, University of Tromsø, Tromsø, 9037
E-mail address: `perj@math.uit.no`