Dozen definitions of the Nijenhuis tensor $N_J \in \Lambda^2 T^*M \otimes TM$ of an almost complex structure $J \in T^*M \otimes TM$.

This tensor is an obstruction for an almost complex structure to origin from the complex structure.

1. $N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y]$, $X,Y \in TM$, where the right hand side is calculated for arbitrary vector fields X,Y with the given values $X_a, Y_a \in T_aM$ at the point $a \in M$. In coordinates it has the formula [NN], [NW]: $N^i_{jk} = \sum J^l_j \partial_l J^i_k - J^l_k \partial_l J^i_j - J^i_l \partial_j J^l_k + J^i_l \partial_k J^l_j$.

2. J-antilinear by each argument part of the torsion of any almost complex connection ∇ (i.e. such a connection that $\nabla J = 0$) $N_J = 4T^{--}_\nabla$. In other words, $N_J(X,Y) = T_\nabla(X,Y) + JT_\nabla(JX,Y) + JT_\nabla(X,JY) - T_\nabla(JX,JY)$. There are connections ∇ called minimal such that $N_J = 4T_\nabla$. [Li].

3. Nijenhuis-Frölicher bracket (differential concomitant) $N_J = [\![J,J]\!]$ of the vector valued 1-form J with itself. [FN].

4. Let $\bar\partial : C^\infty(\Lambda^{p,q}T^*M) \to C^\infty(\Lambda^{p,q+1}T^*M)$ be the component of the de Rham differential. The Nijenhuis tensor is the only obstruction for the Dolbeault sequence to be a complex [Hö]:
$$\dots \xrightarrow{\bar\partial} C^\infty(\Lambda^{p,q}T^*M) \xrightarrow{\bar\partial} C^\infty(\Lambda^{p,q+1}T^*M) \xrightarrow{\bar\partial} \dots$$

5. Structure function of the first order C_G for the G-structure with $G = GL_{\mathbb{C}}(n)$ associated with the almost complex structure J. [St].

6. Weyl tensor $W_1(\mathcal{E}_G) \in H^{0,2}(\mathcal{E}_G)$ of the homogeneous PDE (geometrical structure) modeled on the affine complex space $\mathbb{C}^n$; the group $H^{0,2}(\mathcal{E}_G)$ is the second Spencer cohomology group. [KL].

7. $N_J(X,Y) = (\nabla_X J)(JY) - (\nabla_Y J)(JX) + (\nabla_{JX} J)(Y) - (\nabla_{JY} J)(X)$, where ∇ is any symmetric connection on M. [K1].

8. Let g be a compatible metric, i.e. $\Omega(X,Y) = g(X,JY)$ is a 2-form. Then the Nijenhuis tensor can be found from the following formula, where ∇ is the Levi-Civita connection of g (hence symmetric, see 7) [KN]:
$g(N_J(X,Y),Z) = 3d\Omega(JX,JY,JZ) - 3d\Omega(X,Y,JZ) - 2g((\nabla_{JZ}J)X,Y)$.

9. Let $\Theta(X,Y) = \Omega(JX,Y)$ be 2-form (not metric as in 8). Then we can define the tensor by the formula $\Theta(N_J(X,Y),Z) = d\Theta(X,Y,Z) - d\Theta(JX,JY,Z)$. In the case when $\dim M = 4$ we can divide $d\Theta = \Omega \wedge \alpha$ and $N_J(X,Y) = \alpha(JY)X - \alpha(JX)Y + \alpha(Y)JX - \alpha(X)JY - 2\Omega(X,Y)JX_\alpha$, where $\Theta(X_\alpha, Z) = \alpha(Z)$. [K2]

10. The second generator of the invariant tensor algebra $\langle J, N_J \rangle$ describing the image of the projection $\pi_{2,1} : J^2_{PH}(M,M) \to J^1_{PH}(M,M)$ of pseudo-holomorphic jets. [K1].

11. The real part of the curvature of the distribution $\mathcal{D}$ generated by the projector of the complexified space $P = J+i : T^{\mathbb{C}}M \to T^{\mathbb{C}}M$; $\mathcal{D} = \mathrm{Im}(P)$. Hence $N_J \in \Lambda^{2,0}(T^{\mathbb{C}}M)^* \otimes T^{\mathbb{C}}_{0,1}M$. [KN], [Ko].

12. The homomorphism $d_0^{2,-1} : J_1 \to \Lambda^2(D_J^*)$ for non-holonomic filtration of the projective module determined by the module J_1. In the almost complex case $J_1 = \Lambda^{1,0}(T^{\mathbb{C}}M)^*$. [LR].

B.K.

References.

FN A.Frolicher, A.Nijenhuis "*Theory of vector-valued differential forms*" (I), Proc. Koninkl. Nederl. Akad. Wetensch., ser.A, **59**, issue 3 (1956), 338-359.

Hö L.Hörmander, "*The Frobenius-Nirenberg theorem*", Arkiv for Mathematik **5** (1964), 425-432.

KL B.Kruglikov, V.Lychagin "*On equivalence of differential equations*", Acta et Commentationes Universitatis Tartuensis de Matematica, **3** (1999), 7–29

KN S.Kobayashi, K.Nomizu "*Foundations of Differential Geometry*" II, Wiley-Interscience (1969).

Ko J.J.Kohn, "*Harmonic integrals on strongly pseudo-convex manifolds*" (I), Ann. of Math. **78** (1963), 206-213.

K1 B.Kruglikov "*Nijenhuis tensors and obstructions for pseudoholomorphic mapping constructions*", Mathematical Notes **63**, issue 4 (1998), 541-561.

K2 B.Kruglikov, "*Some classificational problems in four dimensional geometry: distributions, almost complex structures and Monge-Ampere equations*", Math. Sbornik, **189**, no. 11 (1998), 61-74.

Li A.Lichnerowicz "*Theorie globale des connexions et des groupes d'holonomie*", Roma, Edizioni Cremonese (1955).

LR V.Lychagin, V.Rubtsov "*Non-holonomic filtrations: Algebraic and geometric aspects of non-integrability*", Geometry in Partial differential equations, Ed. A.Prastaro, Th.M.Rassias (1994), 189-214.

NN A.Newlander, L.Nirenberg "*Complex analytic coordinates in almost-complex manifolds*", Ann. Math. **65**, ser. 2, issue 3 (1957), 391-404.

NW A.Nijenhuis, W.Wolf "*Some integration problems in almost-complex and complex manifolds*", Ann. Math. **77** (1963), 424-489.

St S. Sternberg "*Lectures on differential geometry*", Prentice-Hall, New Jersey (1964).